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Approximate Solution to the Schrödinger Equation with Manning-Rosen plus a Class of Yukawa Potential via WKBJ Approximation Method

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Abstract. Exact analytical investigations of the Schrödinger equation for pure and mixed potentials have been of significance interests in recent years. Here, we have solved and obtained the exact eigenenergy solutions for Manning–Rosen plus a class of Yukawa potential via the Wentzel–Kramers–Brillouin and Jeffery (WKBJ) approach. The energy eigenvalues and the corresponding wave functions of the system are also obtained analytically. The results investigated for the potential are in agreement with those obtained by the other theoretical methods.

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1 Introduction

Exact solutions to equations play an important role in theoretical physics because they possessed a wealth of vital information regarding the system under consideration. For instance, the exact analytical eigensolutions of the Schrödinger equation for the hydrogen atom and simple harmonic oscillator provided strong evidence supporting the significance of the quantum theory. However many quantum systems are treated as approximations because exact solutions are few [1–4]. The bound state energy equation and the unnormalized radial wave functions have been approximately obtained for the Manning-Rosen potential by using the supersymmetric WKB approach and the function analysis method

[5–7]. The exact analytical eigenstate solutions of the Dirac equation using the Manning–Rosen potential for an arbitrary spin-orbit coupling quantum have been obtained [8].

Wentzel, Kramers, Brillouin and Jeffery also known generally as the WKB approximation method is one of the simplest and advanced methods of obtaining estimate eigenvalues of the one-dimensional Schrödinger equation in the limiting case of large quantum numbers was initially proposed by [9–12]. In the lowest order approximation, the WKB quantization condition is

$$\int_{r_1}^{r_2} \sqrt{2m\,(E - V(r))}dr = \pi\hbar\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots \tag{1}$$

In general, Eq. (1) yields averagely accurate eigenvalues as analytical exact functions of the parameters contained in the potential.

The WKB approximation is adequately use for three-dimensional problems with spherical symmetry by applying the one-dimensional WKB formalism to the radial Schrödinger equation given as:

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{2m}{\hbar^2}\left[E - V_{\text{eff}}\,(r)\right]\Psi = 0\,, \tag{2}$$

where the effective potential $V_{\text{eff}}\,(r)$ is

$$V_{\text{eff}}\,(r) = V\,(r) + \frac{l(l+1)\hbar^2}{2mr^2}\,.$$

Such a direct utilization results to a paramount challenge in obtaining exact analytical energy eigenvalue solutions because the WKB reduced radial wave function at the origin has a response which is contrasting from that of the actual wave function [13]. For this reason, Langer [14] proposed that the robustness of the angular momentum $l(l+1)$ should be viewed as an adjustable parameter K, not as a fixed quantity. Langer suggests out that K should be swapped with the term $\left(l + \frac{1}{2}\right)^2$ in the lowest order quantization formula which has great physical meaning. The substitution of $l\,(l+1) \to \left(l + \frac{1}{2}\right)^2$ regularizes the radial WKB wave function at the origin and ensure adequate asymptotic behaviour at large quantum numbers [15, 16].

In this work, our purpose is to solve the Schrödinger equation for the Manning-Rosen via the WKB approximation method. The Manning-Rosen plus a class of Yukawa potential takes the form:

$$V\,(r) = -\left[\frac{Ce^{-\alpha r} + De^{-2\alpha r}}{\left(1 - e^{-\alpha r}\right)^2}\right] - \frac{V_0}{r} - \frac{V_1 e^{-\alpha r}}{r} - \frac{V_2 e^{-2\alpha r}}{r^2} \tag{3}$$

where α is the screening parameter, C, D and $V_0 V_1, V_2$ are the depth of the potential. Not much has been done in solving the Manning-Rosen plus a class of Yukawa potential via the WKB method.

This paper is organized as follows: Section 1 contains the introduction, a brief illustration of the semiclassical quantization and the WKB approximation method for the radial solution is reviewed in Section 2. In Section 3, the radial Schrödinger equation with Manning-Rosen plus a class of Yukawa potential is solved. Finally, we give a brief discussion in Section 4 before the conclusion in Section 5.

2 Semiclassical Quantization and the WKB Approximation

In this section, we consider the quasi-classical solution of the Schrödinger's equation for the spherically symmetric potentials. Given the Schrödinger equation for a spherically symmetric potentials $V(r)$ of Eq. (3) as

$$(-i\hbar)^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\psi(r,\theta,\phi) = [2m(E-V(r))]\psi(r,\theta,\phi)\,. \quad (4)$$

The total wave function in Eq. (3) can be defined as

$$\psi(r\theta\phi)=[rR(r)]\left[\sqrt{\sin\theta}\Theta(\theta)\Phi(\phi)\right]. \quad (5)$$

And by decomposing the spherical wave function in Eq. (4) using Eq. (5) we obtain the following equations:

$$\left(-i\hbar\frac{d}{dr}\right)^2 R(r)=\left[2m(E-V(r))-\frac{\vec{M}^2}{r^2}\right]R(r)\,, \quad (6)$$

$$\left(-i\hbar\frac{d}{d\theta}\right)^2\Theta(\theta)=\left[\vec{M}^2-\frac{M_z^2}{\sin^2\theta}\right]\Theta(\theta)\,, \quad (7)$$

$$\left(-i\hbar\frac{d}{d\phi}\right)^2\Phi(\phi)=M_z^2\Phi(\phi)\,, \quad (8)$$

where $\vec{M}^2$, M_z^2 are te constants of separation and, at the same time, integrals of motion. the squared angular momentum $\vec{M}^2=\left(l+\frac{1}{2}\right)^2\hbar^2$.

Considering Eq. (6), the leading order WKB quantization condition appropriate to Eq. (3) is

$$\int_{r_1}^{r_2}\sqrt{P^2(r)}dr=\pi\hbar\left(n+\frac{1}{2}\right),\quad n=0,1,2,\ldots, \quad (9)$$

where r_2 and r_1 are the classical turning point known as the roots of the equation

$$P^2(r)=2m(E-V(r))-\frac{\left(l+\frac{1}{2}\right)^2\hbar^2}{r^2}=0\,. \quad (10)$$

Equation (9) is the WKB quantization condition which is subject to discussion in the preceding section. Consider Eq. (6)–(8) in the framework of the quasi-classical method, the solution of each of these equations in the leading $\hbar$ approximation can be written in the form

$$\Psi^{WKB}(r) = \frac{A}{\sqrt{P(r,\lambda)}} exp\left[\pm\frac{i}{\hbar}\int\sqrt{P^2(r)}dr\right]. \tag{11}$$

3 Solutions to the Radial Schrödinger Equation

The radial Schrödinger equation for the Manning-Rosen plus a class of Yukawa potential can be obtained numerically using the WKB quantization condition Eq. (9). Since the potential of interest gradually varies, we presume that the wave function remains sinusoidal. Hence, we apply the effective potential and plug it into the WKB approximation of Eq. (10) and to obtain the exact solution, we consider two turning points.

Given the effective potential with the centrifugal term as

$$V_{\text{eff}}(r) = -\left[\frac{Ce^{-\alpha r} + De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] - \frac{V_0}{r} - \frac{V_1 e^{-\alpha r}}{r} - \frac{V_2 e^{-2\alpha r}}{r^2} + \frac{\left(l+\frac{1}{2}\right)^2\hbar^2}{2mr^2} \tag{12}$$

The wave equation (12) is not an exactly solvable problem even for $l = 0$ because of the centrifugal barrier term. Therefore, to solve Eq. (12) analytically, we use an approximation scheme of the exponential-type proposed by Greene and Aldrich [12] to deal with the centrifugal term:

$$\frac{1}{r^2} = \frac{\alpha^2}{\left(1-e^{-\alpha r}\right)^2}. \tag{13}$$

Equation (13) is a good approximation to the centrifugal/inverse square term in a short potential range.This approximation is valid when $\alpha \ll 1$ [17]. The potential in Eq. (12) can also be written in the form:

$$V_{\text{eff}}(r) = -\frac{Ce^{-\alpha r}}{(1-e^{-\alpha r})^2} - \frac{De^{-2\alpha r}}{(1-e^{-\alpha r})^2} - \frac{V_0\alpha}{1-e^{-\alpha r}} - \frac{V_1\alpha e^{-\alpha r}}{1-e^{-\alpha r}} - \frac{V_2\alpha^2 e^{-2\alpha r}}{\left(1-e^{-\alpha r}\right)^2} + \frac{\alpha^2\hbar^2\left(l+\frac{1}{2}\right)^2 e^{-\alpha r}}{2m\left(1-e^{-\alpha r}\right)^2}. \tag{14}$$

Substituting Eq. (14) into Eq. (9), we have

$$\int_{r_1}^{r_2} \sqrt{P^2(r)}dr$$
$$= \int_{r_1}^{r_2} \left\{ 2m\Big(E_{nl} + \frac{Ce^{-\alpha r}}{(1-e^{-\alpha r})^2} + \frac{De^{-2\alpha r}}{(1-e^{-\alpha r})^2} + \frac{V_0\alpha}{1-e^{-\alpha r}} + \frac{V_1\alpha e^{-\alpha r}}{1-e^{-\alpha r}} \right.$$
$$\left. + \frac{V_2\alpha^2 e^{-2\alpha r}}{(1-e^{-\alpha r})^2} - \frac{\alpha^2\hbar^2\left(l+\frac{1}{2}\right)^2 e^{-\alpha r}}{2m\left(1-e^{-\alpha r}\right)^2} \Big) \right\}^{1/2} dr = \pi\Big(n+\frac{1}{2}\Big). \quad (15)$$

Let

$$\vec{M}^2 = \frac{\alpha^2\hbar^2\left(l+\frac{1}{2}\right)^2}{2m} \quad (16)$$

$$\int_{r_1}^{r_2} \left\{ 2m\Big(E_{nl} + \frac{Ce^{-\alpha r}}{(1-e^{-\alpha r})^2} + \frac{De^{-2\alpha r}}{(1-e^{-\alpha r})^2} \right.$$
$$\left. + \frac{V_0\alpha}{1-e^{-\alpha r}} - \frac{V_1\alpha e^{-\alpha r}}{1-e^{-\alpha r}} - \frac{\vec{M}^2}{(1-e^{-\alpha r})^2} \Big) \right\}^{1/2} dr = \pi\hbar\Big(n+\frac{1}{2}\Big). \quad (17)$$

Making the transformation

$$z = \frac{e^{-\alpha r}}{1-e^{-\alpha r}}, \quad (18)$$

we obtain

$$\frac{-\sqrt{2m}}{\alpha\hbar} \int_{z_1}^{z_2} \frac{1}{z(1+z)} \left\{ E_{nl} + Cz(1+z) + V_0\alpha(1+z) + V_1\alpha z \right.$$
$$\left. + V_2\alpha^2 z + Dz^2 - \vec{M}^2(1+2z+z^2) \right\}^{1/2} dz = \pi\left(n+\frac{1}{2}\right); \quad (19)$$

$$\frac{-\sqrt{2m}}{\alpha\hbar} \int_{z_1}^{z_2} \frac{1}{z(1+z)} \left\{ -(\vec{M}^2 - C - D - V_2\alpha^2)z^2 + (C + V_0\alpha + V_1\alpha - 2\vec{M}^2)z \right.$$
$$\left. + E_{nl} + V_0\alpha - \vec{M}^2 \right\}^{1/2} dz = \pi\left(n+\frac{1}{2}\right); \quad (20)$$

$$\frac{-\sqrt{2m(\vec{M}^2-C-D-V_2\alpha^2)}}{\alpha\hbar}\int\limits_{z_1}^{z_2}\frac{1}{z(1+z)}\left\{-z^2+\frac{C+V_0\alpha+V_1\alpha-2\vec{M}^2}{(\vec{M}^2-C-D-V_2\alpha^2)}z\right.$$
$$\left.+\frac{E_{nl}+V_0\alpha-\vec{M}^2}{(\vec{M}^2-C-D-V_2\alpha^2)}\right\}^{1/2}dz=\pi\left(n+\frac{1}{2}\right). \quad (21)$$

Let

$$\frac{C+V_0\alpha-V_1\alpha-2\vec{M}^2}{(\vec{M}^2-C-D-V_2\alpha^2)}=b \text{ and } \frac{E_{nl}+V_0\alpha-\vec{M}^2}{(\vec{M}^2-C-D-V_2\alpha^2)}=-c, \quad (22)$$

we have

$$\frac{-\sqrt{2m(\vec{M}^2-C-D-V_2\alpha^2)}}{\alpha\hbar}\int\limits_{z_1}^{z_2}\frac{1}{z(1+z)}\sqrt{-z^2+bz-c}dz$$
$$=\pi\left(n+\frac{1}{2}\right); \quad (23)$$

$$\frac{-\sqrt{2m(\vec{M}^2-C-D-V_2\alpha^2)}}{\alpha}\int\limits_{z_1}^{z_2}\frac{1}{z(1+z)}\sqrt{(z-z_1)(z_2-z)}dz$$
$$=\pi\hbar\left(n+\frac{1}{2}\right), \quad (24)$$

where we obtain the turning points z_2 and z_1 from the terms inside the square roots as

$$z_1=\frac{-b-\sqrt{b^2-4C}}{2}; \quad z_2=\frac{-b+\sqrt{b-4C}}{2}.$$

Let

$$2z+1=y; \qquad dz=\frac{dy}{2}. \quad (25)$$

Substituting Eq. (25) into Eq. (24), we obtain

$$\int\limits_{y_1}^{y_2}\frac{1}{y^2-1}\sqrt{(y-y_1)(y_2-y)}dy=\frac{-\alpha\pi\hbar\left(n+\frac{1}{2}\right)}{\sqrt{2m(\vec{M}^2-C-D-V_2\alpha^2)}}. \quad (26)$$

For computing the integral in equation (26), we use the integral expression [13, 14]

$$\int_{y_1}^{y_2}\frac{1}{y^2-1}\sqrt{(y-y_1)(y_2-y)}dy$$
$$=\frac{\pi}{2}\left[\sqrt{(y_1+1)(y_2+1)}-\sqrt{(y_1-1)(y_2-1)}+2\right], \quad (27)$$

where the limits$y_1 y_2$ are real numbers, with $y_1 < y_2$. Comparing equation (27) with equation (26), and solving for E_{nl} gives

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\frac{2\mu V_0}{\alpha\hbar^2} - \left(l+\frac{1}{2}\right)^2\left[2\left(l+\frac{1}{2}\right)^2 - \frac{2\mu C}{\alpha^2\hbar^2} + \left(n+\frac{1}{2}\right)^2 - \frac{2\mu V_0}{\alpha\hbar^2}\right.\right.$$
$$\left. - \frac{2\mu V_1}{\alpha\hbar^2} + (2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{2\mu C}{\alpha^2\hbar^2} - \frac{2\mu D}{\alpha^2\hbar^2} - \frac{2\mu V_2}{\hbar^2}}\right]^2$$
$$\left.\times\left(2n+1+2\sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{2\mu C}{\alpha^2\hbar^2} - \frac{2\mu D}{\alpha^2\hbar^2} - \frac{2\mu V_2}{\hbar^2}}\right)^{-2}.\right\} \tag{28}$$

4 Discussions

Case 1 : If $V_0 = V_1 = V_2 = 0$ in Eq. (3), we obtain the energy equation of Manning–Rosen potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[2\left(l+\frac{1}{2}\right)^2 - \frac{2\mu C}{\alpha^2\hbar^2} + \left(n+\frac{1}{2}\right)^2\right.\right.$$
$$\left. + (2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{2\mu C}{\alpha^2\hbar^2} - \frac{2\mu D}{\alpha^2\hbar^2}}\right]^2$$
$$\left.\times\left[2n+1+2\sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{2\mu D}{\alpha^2\hbar^2} - \frac{2\mu C}{\alpha^2\hbar^2}}\right]^{-2} - \left(l+\frac{1}{2}\right)^2\right\}. \tag{29}$$

Case 2 : If $C = D = 0$ in Eq. (3), we obtain the energy equation of the Class of Yukawa potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\frac{2\mu V_0}{\alpha\hbar^2} - \left(l+\frac{1}{2}\right)^2\left[2\left(l+\frac{1}{2}\right)^2 + \left(n+\frac{1}{2}\right)^2 - \frac{2\mu V_0}{\alpha\hbar^2}\right.\right.$$
$$\left. - \frac{2\mu V_1}{\alpha\hbar^2} + (2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{2\mu V_2}{\hbar^2}}\right]^2$$
$$\left.\times\left[2n+1+2\sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{2\mu V_2}{\hbar^2}}\right]^{-2}\right\}. \tag{30}$$

Case 3 : If $V_1 = V_2 = 0$, $C = D = 0$ in Eq. (3), we obtain the energy equation of the coulomb potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\Bigg\{\frac{2\mu V_0}{\alpha\hbar^2} - \left(l+\frac{1}{2}\right)^2\Bigg[2\left(l+\frac{1}{2}\right)^2 + \left(n+\frac{1}{2}\right)^2 - \frac{2\mu V_0}{\alpha\hbar^2} + (2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2}\Bigg]^2\Bigg[2(n+l+1)\Bigg]^{-2}\Bigg\}. \quad (31)$$

Case 4 : If $V_0 = V_2 = 0$, $C = D = 0$ in Eq. (3), we obtain the energy equation of the Yukawa potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\Bigg\{\Bigg[2\left(l+\frac{1}{2}\right)^2 + \left(n+\frac{1}{2}\right)^2 - \frac{2\mu V_1}{\alpha\hbar^2} + (2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2}\Bigg]^2\Bigg[2(n+l+1)\Bigg]^{-2} - \left(l+\frac{1}{2}\right)^2\Bigg\}. \quad (32)$$

Case 5 : If $V_1 = -V_1$, $V_2 = 0$, $C = D = 0$ in Eq. (3), we obtain the energy equation of the Hellmann potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\Bigg\{\frac{2\mu V_0}{\alpha\hbar^2} - \left(l+\frac{1}{2}\right)^2\Bigg[2\left(l+\frac{1}{2}\right)^2 + \left(n+\frac{1}{2}\right)^2 - \frac{2\mu V_0}{\alpha\hbar^2} + \frac{2\mu V_1}{\alpha\hbar^2} + (2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2}\Bigg]^2\Bigg[2(n+l+1)\Bigg]^{-2}\Bigg\}. \quad (33)$$

Table 1. Bound state energy eigenvalues for $\alpha = 0.001$ and $\alpha = 0.075$ for different states of the mixed potential (MRCYP) atomic unit $\hbar = \mu = 1$; $V_0 = 0.25$ MeV, $V_1 = 2.725$ MeV, $V_2 = 1.075$ MeV, $C = 0.75$ MeV, $D = -1.325$ MeV

	$\alpha = 0.001$		$\alpha = 0.075$	
n, l	WKB	NU	WKB	NU
1,0	-0.24502133	-0.24502153	-0.27468534	-0.27581559
2,0	-0.24386790	-0.24386811	-0.19856089	-0.19960083
2,1	-0.24386518	-0.24386538	-0.18567024	-0.18670197
3,0	-0.24271869	-0.24271889	-0.14011069	-0.14107144
3,1	-0.24271597	-0.24271618	-0.12891193	-0.12986939
3,2	-0.24271054	-0.24271074	-0.10705402	-0.10799775
4,0	-0.24157368	-0.24157388	-0.09594498	-0.09684578
4,1	-2.24157097	-0.24157117	-0.08615093	-0.08704564
4,2	-0.24156555	-0.24156575	-0.06700668	-0.06788953
4,3	-0.24155741	-0.24155762	-0.03934792	-0.04021363

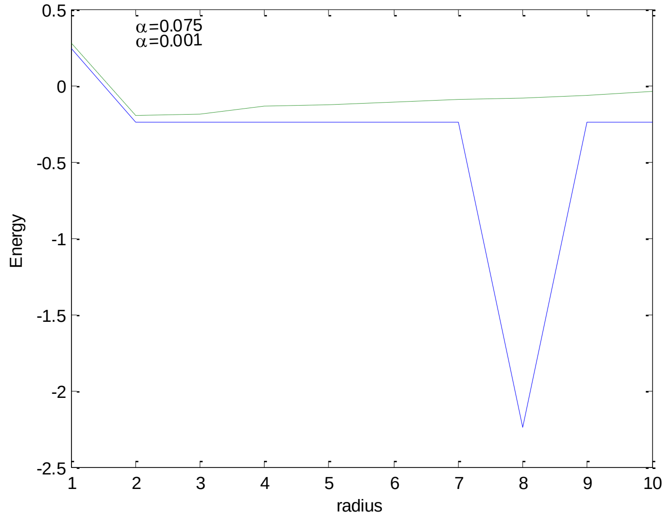

Figure 1. (Color online) Variation of the energy spectrum as a function of the radius for different values of the screening parameter for $\alpha = 0.001$ and $\alpha = 0.075$ for different States of the mixed potential (MRCYP) atomic unit $\hbar = \mu = 1$; $V_0 = 0.25$ MeV, $V_1 = 2.725$ MeV, $V_2 = 1.075$ MeV.

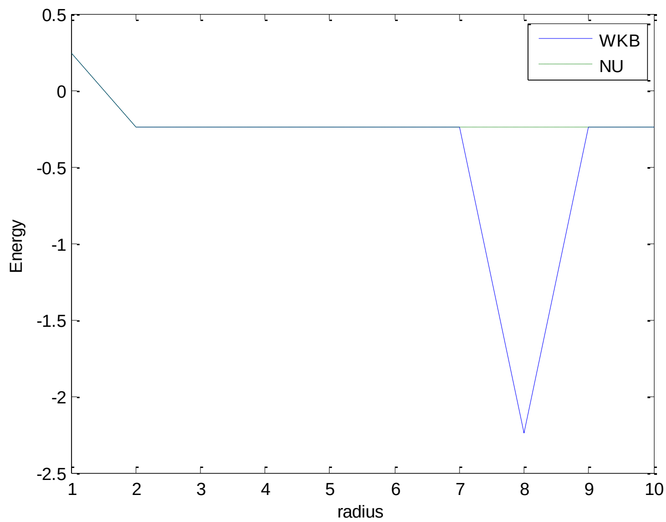

Figure 2. (Color online) Comparative plot of variation of the energy spectrum (for NU and WKB) as a function of the radius for the screening parameter $\alpha = 0.001$ for different states of the mixed potential (MRCYP) atomic unit $\hbar = \mu = 1$; $V_0 = 0.25$ MeV, $V_1 = 2.725$ MeV, $V_2 = 1.075$ MeV.

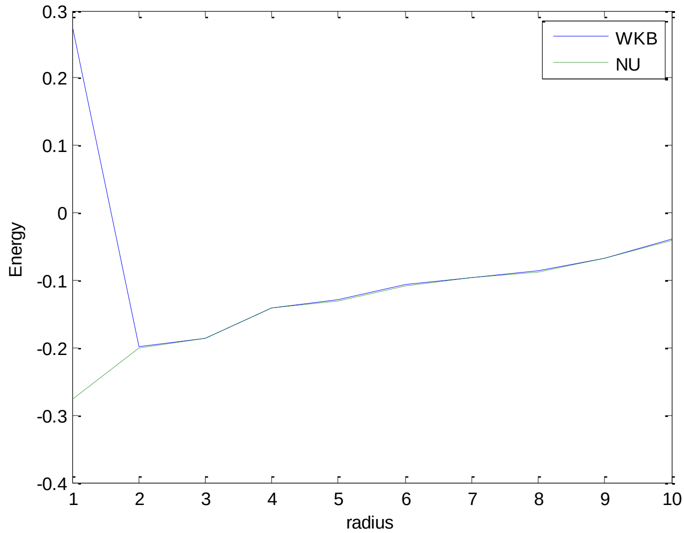

Figure 3. (Color online) Comparative plot of variation of the energy spectrum (for NU and WKB) as a function of the radius for the screening parameter $\alpha = 0.075$ for different states of the mixed potential (MRCYP) atomic unit $\hbar = \mu = 1$; $V_0 = 0.25$ MeV, $V_1 = 2.725$ MeV, $V_2 = 1.075$ MeV.

5 Conclusion

In this paper, we have solved and obtained the exact analytical energy spectrum for Manning–Rosen plus a class of Yukawa potential via the Wentzel–Kramers–Brillouin and Jeffery (WKBJ) approach. Both the wave functions and the corresponding energy spectra of the system derived have exact and explicit forms. Some remarkable results are obtained and noted. By various choices of potential parameters considerations, our results can also be used to deduce the eigen solutions of some quantum mechanical systems. In Figures 1 – 3, we make some plots to see the behavior of energy with screening parameter, potential strength and particle mass.

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