Bulg. J. Phys. **46** (2019) 163–173

Ricci ρ-Soliton and Geometrical Structure in a Dust Fluid and Viscous Fluid Spacetime

M.D. Siddiqi
Department of Mathematics Faculty of Science, Jazan University,
Jazan Kingdom of Saudi Arabia; e-mail: msiddiqi@jazanu.edu.sa

Received: *26 March 2019*

Abstract. he present discourse is based on the geometrical bearing of dust fluid and viscous fluid spacetime in terms of Ricci ρ-soliton and η-Ricci ρ-soliton in viscous fluid and dust fluid spacetime with torse-forming vector field ξ. A condition for the Ricci ρ-soliton to be steady, expanding or shrinking is also given. In particular case when the potential vector filed ξ of the soliton is of gradient type, we derive, from the η-Ricci ρ-soliton equation, a Laplacian equation.

PACS codes: 02.40.-k

1 Introduction

General Relativity is the theory of gravity put forward by Albert Einstein in 1915. In this theory the gravitational field is the spacetime curvature and its source is energy-momentum tensor. This is the origin of all field theories. In general relativity the matter content of the universe is assumed to behave like a perfect fluid in the standard cosmological models, the physical motivation for studying Lorentzian manifolds is the assumption that a gravitational field may be effectively modeled by some Lorentzian metric defined on a suitable 4-dimensional manifold M.

The Einstein's equations are fundamental in the construction of cosmological models which imply that the matter determines the geometry of the spacetime and conversely the motion of matter is determined by the metric tensor of the space which is non-flat. Relativistic fluid models are of considerable interest in several areas of astrophysics, plasma physics and nuclear physics. Theories of relativistic stars (which would be models for supermassive stars) are also based on relativistic fluid models. The problem of accretion onto a neutron stars or a black hole is usually set in the framework of relativistic fluid models.

A connected 4-dimensional Lorentzian manifold is a special subclass of pseudo-Riemannian manifolds with Lorentzian metric g with signature $(-,+,+,+)$

has great importance in general relativity. The geometry of 4-dimensional Lorentzian manifold begins with the study of nature of vectors on the manifold. Therefore, 4-dimensional Lorentzian manifold becomes most suitable choice for the study of general relativity.

A perfect fluid to be one with no heat conduction and no viscosity or it can be defined as fluid which looks isotropic or star in its rest frame. The most simple example of the perfect fluid is dust. Perfect fluid are often used in general relativity to model idealized distribution.

Liquids have molecules that are in contact but are capable of sliding over one another effortlessly. No shear stresses can exist in such a perfect fluid. Water, the most common liquid of all, has properties quite close to those of a perfect fluid. Viscous fluids consist of molecules that like those of the perfect fluid. Thus, fluids of which the viscosity, or internal friction, must be taken into account are called viscous fluids and are further distinguished as *Newtonian fluids* if the viscosity is constant for different rates of shear and does not change with time [14].

The energy-momentum tensor plays the major role as a matter content of the spacetime, matter is assumed to be fluid having density, pressure and having dynamical and kinematic quantities like velocity, acceleration, vorticity, shear and expansion [16]. The matter content of the universe is assumed to perform like a perfect fluid in standard cosmological models.

Thus, the general form of the energy-momentum tensor T for a dust or pressure-less fluid spacetime is [18]

$$T(X,Y) = \sigma\eta(X)\eta(Y) \tag{1}$$

for any $X, Y \in \chi(M)$, where σ is the energy-density, g is the metric tensor of *Minkowski spacetime*, $\eta(X) = g(X, \xi)$ is 1-form, equivalent to the velocity vector of the fluid ξ and $g(\xi, \xi) = -1$.

Let us consider the energy-momentum tensor T of a viscous fluid spacetime in the following form ([13, 14]):

$$T(X,Y) = pg(X,Y) + (\sigma + p)\eta(X)\eta(Y) + P(X,Y)\,, \tag{2}$$

where σ, p are the energy density and isotropic pressure respectively and P denotes the an isotropic pressure tensor to the viscous fluid.

Further, example of energy-momentum tensor are energy-momentum tensor of electromagnetism and scalar field theory.

The field equation governing the perfect fluid motion is Einstein's gravitational equation [14],

$$\kappa T(X,Y) = S(X,Y) + \left(\lambda - \frac{r}{2}\right) g(X,Y), \tag{3}$$

for any $X, Y \in \chi(M)$, where λ is the cosmological constant, κ is the gravitational constant (which can be taken $8\pi G$, with G the universal gravitational constant), S is the Ricci tensor and r is the scalar curvature of g. They are obtained from Einstein's equations by adding a cosmological constant in order to get a static universe, according to Einstein's idea. In modern cosmology, it is considered as a candidate for dark energy, the cause of the acceleration of the expansion of the universe.

From equations (1) and (3) we obtain the Einstein's equation for dust fluid

$$S(X,Y) = -\left(\lambda - \frac{r}{2} + \kappa\rho\right) g(X,Y) + \kappa(\sigma+\rho)\eta(X)\eta(Y). \tag{4}$$

Also, from equations (1) and (4) we obtain the Einstein's equation for viscous fluid

$$S(X,Y) = -\left(\lambda - \frac{r}{2} + \kappa p\right) g(X,Y) + \kappa(\sigma+p)\eta(X)\eta(Y) + \kappa P(X,Y). \tag{5}$$

By the the property of the manifold that the Ricci tensor S is a functional combination of g and $\eta \otimes \eta$, for η a 1-form g dual to a unitary vector field, is called quasi-Einstein ([8, 9]). Perfect fluid spacetime are extensively studied in many manners of views, we may refer to (see [2, 7, 10, 12, 15]) and references therein. In the papers also see ([1, 3, 4, 17, 19]), Ricci solitons are studied extensively within the background of pseudo-Riemannian geometry.

On the other hand in 1981, the notion of Ricci-Bourguignon flow as a generalization of Ricci flow [11] has been introduced by J. P. Bourguinon [5]. Ricci-Bourguignon flow are intrinsec geomeric flow on pseudo-Riemannian manifolds, whose fixed points are solitons.

Ricci ρ-solitons, which generates self-similar solution to the Ricci-Bourguignon flow [6]

$$\frac{\partial g}{\partial t} = -2(S - \rho R g), \quad g(0) = g_0, \tag{6}$$

where S is the Ricci curvature tensor , R is the scalar curvature with respect to the g and ρ is a real non zero constant. It should be noticed that for special values of the constant ρ in equation (6) we have obtain the following situations for the tensor $S - \rho R g$ appearing in equation (6). This PDE system (6) defined the evolution equation is of special interest, in particular [6],

1. $\rho = \frac{1}{2}$, the Einstein tensor $S - \frac{R}{2}g$,
2. $\rho = \frac{1}{n}$, the traceless Ricci tensor $S - \frac{R}{n}g$,
3. $\rho = \frac{1}{2(n-1)}$, the Schoute tenosr $S - \frac{R}{2(n-1)}g$,
4. $\rho = 0$, the Ricci tensor S.

In dimension two, the first three tensors are zero, hence the flow is static and in higher dimension the value of ρ are strictly ordered as above in descending order.

Short time existence and uniqueness for the solution of this geometric flow has been proved in [6]. In fact, for sufficiently small t the equation has a unique solution for $\rho < 1(2(n-1))$.

In the other hand, quasi Einstein metrics or Ricci solitons serve as a solution to Ricci flow equation. This motivates a more general type of Ricci soliton by considering the Ricci-Bourguignon flow. In fact, a pseudo-Riemannian manifold of dimension $n \geq 3$ is said to be Ricci ρ-soliton if

$$\mathcal{L}_V g(X,Y) + 2S(X,Y) + 2(\mu + \rho R)g(X,Y) = 0\,, \tag{7}$$

where, $\mathcal{L}_V$ denotes the Lie derivative operator along vector field V and μ is an arbitrary real constant. Similar to Ricci solitons, a Ricci ρ soliton is called expanding if $\mu > 0$, steady if $\mu = 0$ and shrinking if $\mu < 0$.

Perturbing the equation that define (7) Ricci ρ-solitons by multiple of a certain $(0,2)$-tensor field $\eta \otimes \eta$, we obtain slightly more general notion, namely η-Ricci ρ-solitons, which we shall consider in a dust fluid and viscous fluid spacetime, i.e. in a 4-dimensional pseudo-Riemannian manifold M with Lorentzian metric g whose content are dust fluid and viscous fluid.

In this paper, we have studied some geometrical aspect of Ricci ρ-soliton and η-Ricci ρ-soliton in dust fluid and viscous fluid spacetime with torse-forming vector field ξ.

2 Properties of Viscous Fluid and Dust Fluid Spacetime with Torse-Forming Vector Field

Let (M^4, g) be a relativistic viscous fluid spacetime satisfying (6). Contracting (6) and assumed that $g(\xi,\xi) = -1$, we obtain

$$r = 4\lambda + \kappa[(\sigma - 3p) + J], \tag{8}$$

where $J = trace(P)$. Therefore,

$$S(X,Y) = \Big(\lambda + \frac{\kappa(\sigma - p + J)}{2}\Big)g(X,Y) + \kappa(\sigma + p)\eta(X)\eta(Y) + \kappa P(X,Y). \tag{9}$$

Also

$$S(\xi,\xi) = -\lambda + \frac{\kappa}{2}[\sigma + 3p + J + 2I], \tag{10}$$

where $I = P(\xi,\xi)$.

For the dust fluid spacetime equation (9) reduces in to the following form

$$S(X,Y) = \left(\lambda + \frac{\kappa\sigma}{2}\right) g(X,Y) + \kappa\sigma\eta(X)\eta(Y). \tag{11}$$

Example 1. *A radiation fluid ($\sigma = 3p$) has constant scalar curvature r equal to $4\lambda + J$.*

Definition 1. *A vector field ξ is called torse forming if it satisfies [20]*

$$\nabla_X \xi = X + \eta(X)\xi \tag{12}$$

for a vector field X on M^4 and η is an 1-form.

Theorem 1. *On perfect fluid spacetime with torse-forming vector field ξ, the following relation hold [20]*

$$\eta(\nabla_\xi \xi) = 0, \qquad \nabla_\xi \xi = 0, \tag{13}$$
$$(\nabla_X \eta)(Y) = g(X,Y) + \eta(X)\eta(Y), \tag{14}$$
$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{15}$$
$$R(X,\xi)\xi = -X - \eta(X)\xi, \tag{16}$$
$$\eta(R(X,Y)Z) = \eta(X)g(Y,Z) - \eta(Y)g(X,Z), \tag{17}$$
$$(\mathcal{L}_\xi g)(X,Y) = 2[g(X,Y) + \eta(X)\eta(Y)]. \tag{18}$$

Proof. To calculate $(\nabla_X \eta)(Y) = X(\eta(Y)) - \eta(\nabla_X Y) = X(g(Y,\xi)) - g(\nabla_X Y,\xi) = g(Y,\nabla_X \xi) = g(X,Y) + \eta(X)\eta(Y)$. In particular $(\nabla_\xi \eta)(Y) = 0$. The relation (14) can be obtain by (12).

Now, using (12) in $R(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi$ and from direct calculation we have get the relation (15). Additionally (16) and (17) follows from (15). □

3 Ricci ρ-Soliton in Viscous Fluid and Dust Fluid Spacetime

In this section, we study Ricci ρ-soliton structure in a viscous fluid and dust fluid spacetime whose timelike velocity vector field ξ is torse-forming [3].

Now replacing $V = \xi$, equation (7)

$$\mathcal{L}_\xi g(X,Y) + 2S(X,Y) + 2(\mu + \rho R)g(X,Y) = 0. \tag{19}$$

From, (18), we have

$$S(X,Y) + [\mu + 1]\, g(X,Y) + \eta(X)\eta(Y) = 0. \tag{20}$$

Putting $X = Y = \xi$ in (20), we obtain

$$S(\xi,\xi) = (\mu + \rho R). \tag{21}$$

Now, using (10) in the (22), we obtain

$$\mu = \frac{\kappa}{2}[\sigma + 3p + J + 2I] - (\lambda + \rho R) \tag{22}$$

Thus, we have the following results

Theorem 2. *If a viscous fluid spacetime with torse-forming vector field ξ admits a Ricci ρ-soliton (g, ξ, μ), then Ricci ρ-soliton is expending, steady and shrinking according as*

i $\frac{\kappa}{2}(\sigma + 3p + J + 2I) > \lambda + \rho R$,

ii $\frac{\kappa}{2}(\sigma + 3p + J + 2I) = \lambda + \rho R$ *and*

iii $\frac{\kappa}{2}(\sigma + 3p + J + 2I) < \lambda + \rho R$, *respectively.*

Now, for dust fluid spacetime $p = 0$ and $J = I = 0$, then we have following theorem

Theorem 3. *If a dust fluid spacetime with torse-forming vector field ξ admits a Ricci ρ-soliton (g, ξ, μ), then Ricci ρ-soliton is expending, steady and shrinking according as*

i $\frac{\kappa}{2}\sigma > \lambda + \rho R$,

ii $\frac{\kappa}{2}\sigma = \lambda + \rho R$ *and*

iii $\frac{\kappa}{2}\sigma < \lambda + \rho R$, *respectively.*

4 η-Ricci ρ-Soliton in Viscous Fluid and Dust Fluid Spacetime

Consider the equation

$$\mathcal{L}_\xi g + 2S + 2(\mu + \rho R)g + 2\omega\eta \otimes \eta = 0. \tag{23}$$

where g is a pseudo-Riemannian metric, S is the Ricci curvature, ξ is a vector field, η is a 1-form and μ and ω are real constant. The data (g, ξ, μ, ω) which satisfy the equation (23) is said to be a η-Ricci ρ-soliton in M [19]; in particular if $\omega = 0$, (g, ξ, μ) is a Ricci ρ-soliton [4, 17] and its is called *shrinking*, *steady* or *expanding* according as μ is negative, zero or positive, respectively [17].

Writing explicitly the Lie derivative $\mathcal{L}_\xi g$ we get

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \tag{24}$$

and form (23) we obtain

$$S(X, Y) = -(\mu + \rho R)g(X, Y) - \omega\eta(X)\eta(Y) - \frac{1}{2}[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)], \tag{25}$$

for any $X, Y \in \chi(M)$.

Contracting (25) we get

$$r = -(\mu - \rho R)dim(M) + \omega - div(\xi). \quad (26)$$

Let (M^4, g) be a general relativistic viscous fluid spacetime and (g, ξ, μ, ω) be a η-Ricci ρ-soliton in M. From (4) and (25) we obtain

$$\left[\lambda + \frac{\kappa(\sigma - p + J)}{2} + \mu + \rho R\right] g(X,Y) + [\kappa(\sigma + p) + \omega]\eta(X)\eta(Y) + \kappa P(X,Y) + \frac{1}{2}g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0. \quad (27)$$

for any $X, Y \in \chi(M)$.

Consider $\{e_i\}_{1\leq i\leq 4}$ an orthonormal frame field and $\xi = \sum_{i=1}^{4} \xi^i e_i$. We have $\sum_{i=1}^{4} \varepsilon_{ii}(\xi^i)^2 = -1$ and $\eta(e_i) = \varepsilon_{ii}\xi^i$.

Multiplying (27) by ε_{ii} and summing over i for $X = Y = e_i$, we get

$$4\mu - \omega = -4\lambda - \kappa(\sigma - 3p - J) - \rho R - div(\xi). \quad (28)$$

Writing (27) for $X = Y = \xi$, we obtain

$$\mu - \omega = -\lambda + \frac{\kappa}{2}[\sigma - 3p_J + I] - \rho R \quad (29)$$

Therefore,

$$\begin{cases} \mu = \lambda + \dfrac{\kappa}{2}\Big(\dfrac{\sigma}{3} - 3p - J + \dfrac{I}{3}\Big) - \rho R - \dfrac{div(\xi)}{3} \\ \omega = \kappa\Big(\dfrac{2}{3}\sigma - 3p - J + \dfrac{2}{3}I\Big) - \dfrac{div(\xi)}{3} \end{cases} \quad (30)$$

If M admits the η-Ricci ρ-soliton for dust fluid spacetime, then we obtain

$$\begin{cases} \mu = \lambda + \dfrac{\kappa\sigma}{6} - \rho R - \dfrac{div(\xi)}{3} \\ \omega = \dfrac{2}{3}\kappa\sigma - \dfrac{div(\xi)}{3} \end{cases} \quad (31)$$

Using (30) and (31) we can state the following results

Theorem 4. *Let (M^4, g) be a 4-dimensional pseudo-Riemannaian manifold and η be the g-dual 1-form of the gradient vector field $\xi = grad(\psi)$ with $g(\xi, \xi) = -1$. If (23) define a η-Ricci ρ-soliton for viscous fluid spacetime in M^4, then the Laplacian equation for viscous fluid spacetime, satisfied by ψ becomes*

$$\Delta(\psi) = -3\left[\omega - \kappa\left\{\frac{2}{3}\sigma - 3p - J + \frac{2}{3}I\right\}\right]. \quad (32)$$

Theorem 5. *Let (M^4, g) be a 4-dimensional pseudo-Riemannaian manifold and η be the g-dual 1-form of the gradient vector field $\xi = grad(\psi)$ with $g(\xi, \xi) = -1$. If dust fluid spacetime M^4 admitting the η-Ricci ρ-soliton, then the Laplacian equation satisfied by ψ becomes*

$$\Delta(\psi) = -3\left[\omega - \kappa\left\{\frac{2}{3}\sigma\right\}\right]. \tag{33}$$

Remark 1. *If $\omega = 0$ in (23), we obtain the Ricci ρ-soliton for viscous fluid with $\mu = -\lambda + \frac{\kappa}{2}[\sigma - 3p_J + I] - \rho R$ which is steady if $p = \frac{2}{3}\left(\frac{\lambda}{\kappa}\right) - \frac{\sigma}{3} + \frac{J}{3} + \frac{\rho R}{3}$, expanding if $p > \frac{2}{3}\left(\frac{\lambda}{\kappa}\right) - \frac{\sigma}{3} + \frac{J}{3} + \frac{\rho R}{3}$ and shrinking if $p < \frac{2}{3}\left(\frac{\lambda}{\kappa}\right) - \frac{\sigma}{3} + \frac{J}{3} + \frac{\rho R}{3}$. In these cases, $div(\xi) = 3\left[\kappa\left\{\frac{2}{3}\sigma - 3p - J + \frac{2}{3}I\right\}\right]$.*

From *Plebanski energy conditions* for viscous fluid we deduce that $\sigma \geq max\left\{-\frac{\lambda}{\kappa}, \frac{\lambda}{2\kappa}\right\}$ for steady case, $\sigma > \frac{\lambda}{2\kappa}$ and $\sigma > -\frac{\lambda}{\kappa}$ for the expanding and shrinking case, respectively.

Example 2. *A η-Ricci ρ-soliton (g, ξ, μ, ω) in a radiation fluid is give by*

$$\begin{cases} \mu = \lambda + \dfrac{\kappa}{2}\Big(\dfrac{\sigma}{3} - 3p - J + \dfrac{I}{3}\Big) - \rho R - \dfrac{div(\xi)}{3} \\ \omega = \kappa\Big(\dfrac{2}{3}\sigma - 3p - J + \dfrac{2}{3}I\Big) - \dfrac{div(\xi)}{3} \end{cases}$$

From this example (3), we deduce that Ricci ρ-soliton in radiation fluid is steady if $p = \frac{\lambda}{3\kappa}$, expanding if $p > \frac{\lambda}{3\kappa}$ and shrinking if $p < \frac{\lambda}{3\kappa}$.

Remark 2. *If the vector field ξ is conformally Killing i.e., $\mathcal{L}_\xi g = \alpha g$ with α a nonzero real number, ten the existence of Ricci ρ-soliton given by (23) for $\omega = 0$, implies the vacuum case. Moreover, the Ricci ρ-soliton is steady if $p = \frac{\lambda}{\kappa} + \frac{\alpha}{2\kappa} - \frac{\sigma}{3} + \frac{J}{3} + \frac{\rho R}{3}$, expanding if $p > \frac{\lambda}{\kappa} + \frac{\alpha}{2\kappa} - \frac{\sigma}{3} + \frac{J}{3} + \frac{\rho R}{3}$ and shrinking if $p < \frac{\lambda}{\kappa} + \frac{\alpha}{2\kappa} - \frac{\sigma}{3} + \frac{J}{3} + \frac{\rho R}{3}$.*

5 Example of a 4-Dimensional Lorentzian Manifold Admitting η-Ricci ρ-Soliton

Example 3. *Let 4-dimensional manifold $M = \left\{(x, y, z, t) \in \mathbb{R}^4 : t \neq 0\right\}$ where (x, y, z, t) are the standard coordinates of $\mathbb{R}^4$.*

Let (e_1, e_2, e_3, e_4) be the set of linearly independent vector fields of M, and is defined as

$$e_1 = t\Big(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big), \quad e_2 = t\frac{\partial}{\partial y}, \quad e_3 = t\Big(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\Big), \quad e_4 = (t)^3\frac{\partial}{\partial t}.$$

Let g be the Riemannian metric M defined by

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1, \quad g(e_4,e_4)=-1, \quad g(e_i,e_j)=0,$$

for $i\neq j, i,j=1,2,3,4$.

Let η be the 1-form defined by $\eta(Z)=g(Z,e_4)$ for any $Z\in\chi(M)$.

Also, let φ be the $(1,1)$ tensor field, defined by

$$\varphi(e_1)=e_1, \quad \varphi(e_2)=e_2, \quad \varphi(e_3)=e_3, \quad \varphi(e_4)=0, \quad \xi=(t)^3\frac{\partial}{\partial t}.$$

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have Thus, using the linearity of φ and g, we have

$$[e_1,e_2]=-(t)e_2, \qquad [e_1,e_4]=-(t)^2e_1,$$
$$[e_2,e_4]=-(t)^2e_2, \qquad [e_3,e_4]=-(t)^2e_3.$$

Then for $e_4=\xi$ and using Koszul's formula for the Lorentzian metric g ,we have

$$\nabla_{e_1}e_1=-(t)^2e_4, \quad \nabla_{e_2}e_1=te_2, \quad \nabla_{e_1}e_4=-(t)^2e_1, \quad \nabla_{e_2}e_4=-(t)^2e_2,$$
$$\nabla_{e_3}e_4=-(t)^2e_3, \quad \nabla_{e_3}e_3=-(t)^2e_4, \quad \nabla_{e_2}e_2=-(t)^2e_4-te_1. \tag{34}$$

From (34) we find that the structure (φ,ξ,η,g) is a Lorentzian structure on M. Consequently $M^4(\varphi,\xi,\eta,g)$ is an Lorentzian manifold (4-dimensional Space-time model).

The non-vanishing components of Riemannian curvature and the Ricci tensors are given by

$$R(e_1,e_4)e_1=(t)^4e_4,\ R(e_2,e_4)e_2=(t)^4e_4, \quad R(e_3,e_4)e_3=(t)^4e_4,$$
$$R(e_1,e_3)e_3=(t)^4e_1,\ R(e_1,e_3)e_1=-(t)^4e_3,\ R(e_2,e_3)e_2=-(t)^4e_3,$$
$$R(e_1,e_4)e_4=(t)^4e_1,\ R(e_2,e_4)e_4=(t)^4e_2, \quad R(e_1,e_2)e_2=[(t)^4-(t)^2]e_1,$$
$$R(e_2,e_3)e_3=(t)^4e_2,\ R(e_3,e_4)e_4=(t)^4e_3, \quad R(e_1,e_2)e_1=-[(t)^4-(t)^2]e_2.$$

From the above expression of the curvature tensor we can easily calculate the non-vanishing components of the Ricci tensor S

$$S(e_1,e_1)=3(t)^4-(t)^2, \quad S(e_2,e_2)=3(t)^4-(t)^2$$

similarly we have

$$S(e_3,e_3)=3(t)^4, \quad S(e_4,e_4)=3(t)^4. \tag{35}$$

Therefore,

$$R=S(e_1,e_1)+S(e_2,e_2)+S(e_3,e_3)+S(e_4,e_4)=2[6(t)^4-(t)^2].$$

Now, from equation, (18) and (23), we obtain

$$2[g(e_i, e_i) + \eta(e_i)\eta(e_i)] + 2S(e_i, e_i) + 2(2\mu + \rho R)g(e_i, e_i) + 2\omega\eta(e_i)\eta(e_i) = 0$$

for all $i \in \{1, 2, 3, 4\}$, and we have

$$2[(-1 + \delta_{i4}] + 2S(e_i, e_i) + 2(2\mu + \rho R)g(e_i, e_i) + 2\omega\delta_{i4} = 0$$

for all $i \in \{1, 2, 3, 4\}$, we get $\mu = [(t)^2(1 + \rho) - 3(t)^4(1 + 4\rho) + 1]$ and $\omega = [(t)^2(1 - \rho) - 3(t)^2 + 2]$. Thus the data (g, ξ, μ, ω) is a η-Ricci ρ-soliton on $(M^4, \phi, \xi, \eta, g)$, which is expanding if $(t)^2(1+\rho) > 3(t)^4(1+4\rho)+1$, shrinking if $(t)^2(1 + \rho) < 3(t)^4(1 + 4\rho) + 1$ or steady if $(t)^2(1 + \rho) = 3(t)^4(1 + 4\rho) + 1$.

6 Conclusion

In general theory of relativity, the matter of content of the spacetime is described by choosing the suitable energy-momentum tensor T. Since the matter content of the universe is considered to working like a perfect fluid such as dust fluid and viscous fluid in the standard cosmological models as a connected 4-dimensional Lorentzian manifold. In this framework Einstein's equation play the fundamental role to construct the cosmological model.

The dust fluid spacetime and viscous fluid spacetime manifold modeled as 4-dimensional Lorentzian manifold admitting the Ricci ρ-soliton and also η-Ricci ρ-soliton. Solitons are the natural extension of the Einstein's metric. Therefore, Einstein manifolds arose during the study of exact solution of the Einstein's field equation. We have obtained the condition such as steady, expanding and shrinking for the Ricci ρ-soliton. Further, we generalized the notion of Ricci ρ-soliton called η-Ricci ρ-soliton. Moreover, we have proved that the dust fluid and viscous fluid spacetime admitting the η-Ricci ρ-soliton and satisfies the Laplacian equation with potential vector filed ψ of gradient type.

Acknowledgement

The author is thankful to the referee's for their valuable criticisms, comments and suggestions towards the improvement of the paper.

References

[1] M. Ali, Z. Ahsan (2014) Ricci Solitons and Symmetries of Spacetime Manifold of General Relativity. *Global Journal of Advanced Research on Classical and Modern Geometries* **1**(2) 75-84.

[2] Z. Ahsan, M. Ali (2017) Curvature Tensor for the Spacetime of General Relativity. *Int. J. Geom. Methods Mod. Phys.* **14** 1750078 (13 pages).

[3] S. Azami, G.F. Ramandi (2019) Ricci ρ-solitons on 3-dimensional η-Einstein almost Kenmotsu manifolds. arXiv:1902.08176 [math.DG].

[4] A.M. Blaga (2017) Solitons and geometrical structures in a perfect fluid spacetime. arXiv:1705.04094 [math.DG].

[5] J.P. Bourguignon (1981) Ricci curvature and Einstein metrics, Global differential geometry and global analysis. *Lecture notes in Math.* **838** 42-63.

[6] G. Catino, L. Cremaschi, Z. Djadli, C. Mantegazza, L. Mazzieri (2017) The Ricci-Bourguignon flow, *Pacific J. Math.* **287** 337-370; DOI: 10.2140/pjm.2017.287.337.

[7] M. Chaki, C. Ray (1996) Spacetimes with covariant constant energy momentum tensor. *Int. J. Theor. Phys*. **35**(5) 1027-1032.

[8] U.C. De, G.C. Ghosh (2004) On quasi-Einstein manifolds. *Period. Math. Hungar*. **48**(12) 223-231.

[9] U.C. De, L. Velimirovi (2015) Spacetimes with semisymmetric energy momentum tensor. *Int. J. Theor. Phys.* **54** 1779-1783.

[10] S. Guler, S.A. Demirbag (2016) A study of generalized quasi Einstein spacetime with application in general relativity. *Int. J. Theor. Phys*. **55** 548-562.

[11] R.S. Hamilton (1988) The Ricci flow on surfaces. *Contemporary Mathematics* **71** 237-261.

[12] S. Mallick, U.C. De (2016) Spacetimes with pseudo symmetric energy momentum tensor. *Communications in Physics* **26**(2) 121-128.

[13] M. Novello, M.J. Reboucas (1978) The stability of a rotating universe. *Astrophys. J.* **225** 719-724.

[14] B. O'Neill (1983) "*Semi-Riemannian Geometry with Applications to Relativity*", Academic Press, New York.

[15] S. Ray-Guha (2006) On perfect fluid pseudo Ricci symmetric spacetime. *Tensor (N.S.)* **67** 101-107.

[16] H. Stephani (1982) "*General Relativity-An Introduction to the Theory of Gravitational Field*", Cambridge University Press, Cambridge.

[17] M.D. Siddiqi (2018) Conformal η-Ricci solitons in δ-Lorentzian Trans Sasakian manifolds. *Int. J. Maps Math.* **1** 15-34.

[18] S.K. Srivastava (2008) "*General Relativity and Cosmology*". Prentice-Hall of India Private Limited, New Delhi.

[19] H. Venkatesha, A. Kumara (2018) Ricci soliton and geometrical structure in a perfect fluid space time with torse-forming vector field. arXiv-1806.00599v1[math.DG]. 2018.

[20] K. Yano (1994) On Torse forming direction in a Riemannian space. *Proc. Imp. Acad. Tokyo*. **20** 340-345.