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# A New Simple Algorithm to Determine the Entanglement Status of Multipartite Pure Quantum States

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**Abstract.** One makes use of the well-known Schmidt decomposition theorem to determine the entanglement status of a bipartite pure quantum state. For solving this problem in the bipartite case one performs Schmidt decomposition procedure and obtains Schmidt number. The given bipartite state is entangled if Schmidt number is greater than unity and separable if it is equal to unity. In this paper we extend this idea of applying Schmidt decomposition procedure to a multipartite pure quantum state and develop an easy algorithm to perform complete analysis of the given $N$-partite pure quantum state. We successfully manage the full factorization of the given multipartite pure quantum state and specify the following: Whether the given an $N$-qubit pure quantum state is completely separable ($N$ factors), or completely entangled (no factors), or partially entangled having entangled factors of different sizes which being entangled cannot be factored further. We have thus completely solved the problem of deciding the entanglement status of multipartite pure quantum states and developed an easy method to find all the separable as well as entangled factors of these multipartite pure quantum states.

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## 1 Introduction

One of the central issues in quantum information theory is whether a given multipartite pure quantum state is separable or entangled [1–5]. This important question of deciding whether a given multipartite pure quantum state is separable or entangled is completely solved in this paper. The problem of deciding entanglement status of a bipartite pure quantum state is one of the initial problems encountered in quantum information research and this problem is successfully tackled using the well-known Schmidt decomposition theorem [5–7]. By using Schmidt decomposition theorem and applying Schmidt decomposition procedure one obtains the so called Schmidt number for the given bipartite pure quantum state. Schmidt number is very important property of composite systems which in some sense quantifies the amount of entanglement between these bipartite systems. The value of Schmidt number tells us about entanglement status of given bipartite state, i.e. the given bipartite pure quantum state is entangled if the corresponding Schmidt number is greater than unity and separable when it is equal to unity. Further Schmidt number is invariant under unitary transformation and this algebraic invariance properties makes the Schmidt number a very useful tool [5]. In this paper we extend Schmidt decomposition procedure to $N$-partite pure quantum states and provide complete analysis regarding the entanglement status of these states. This new procedure completely specifies the following: whether the given $N$-partite pure quantum state under consideration is completely separable ($N$ factors), or completely entangled (no factors), or partially entangled having entangled factors of different sizes which cannot be factored further.

Suppose we are given the $N$-qubit pure quantum state $|\psi\rangle$ as follows:

$$|\psi\rangle = \sum_{s=0}^{2^N-1} a_{r_s}|r_s\rangle \tag{1}$$

expressed in terms of the computational basis. Here the basis vectors $|r_s\rangle$ are ordered lexicographically. That is, the corresponding binary sequences are ordered lexicographically: $r_0 = 00\cdots00$, $r_1 = 00\cdots01$, ..., $r_{2^N-1} = 11\cdots11$, so that $|r_0\rangle = |00\cdots00\rangle$, $|r_1\rangle = |00\cdots01\rangle$, ..., $|r_{2^N-1}\rangle = |11\cdots11\rangle$.

Now, let us rewrite this $N$-qubit state as follows: Let $m, n$ be any integers such that $1 \leq m, n < N$ and $m + n = N$. Let the two sets of computational basis vectors ordered lexicographically be $|i_0\rangle = |00\cdots0\rangle, \ldots, |i_{2^m-1}\rangle = |11\cdots1\rangle$ (each of length $m$) and $|j_0\rangle = |00\cdots0\rangle, \ldots, |j_{2^n-1}\rangle = |11\cdots1\rangle$ (each of length $n$). We now rewrite $|\psi\rangle$ thus

$$|\psi\rangle = \sum_{u=0}^{2^m-1}\sum_{v=0}^{2^n-1} a_{i_u j_v}|i_u\rangle \otimes |j_v\rangle. \tag{2}$$

Here in the symbol $a_{i_u j_v}$, the suffix $i_u j_v$ is the juxtaposition of the binary sequences $i_u$ and $j_v$ in that order.

Now using this new form for $|\psi\rangle$ we go ahead with the following important step: We replace $m$-qubit vectors, $|i_u\rangle$, in the first part by 1-qudit vectors, $|u\rangle$, and replace $n$-qubit vectors, $|j_v\rangle$, in the second part by 1-qudit vectors, $|v\rangle$, and further we also replace the binary sequences $i_u$ and $j_v$ by the corresponding labels $u$ and $v$ respectively. Thus, we now have

$$|\psi\rangle = \sum_{u=0}^{2^m-1}\sum_{v=0}^{2^n-1} a_{uv}|u\rangle \otimes |v\rangle. \tag{3}$$

In this way we have represented the given multipartite pure quantum state, $|\psi\rangle$, as a general bipartite state with the basis for the first part is made up of 1-qudit states $|u\rangle$, where $u = 0, 1, 2, \cdots, 2^m - 1$ and the basis for the second part is made up of 1-qudit states $|v\rangle$, where $v = 0, 1, 2, \cdots, 2^n - 1$. This bipartite state belongs to tensor product space, $\mathbb{H}_{\mathbb{A}} \otimes \mathbb{H}_{\mathbb{B}}$, where $\mathbb{H}_{\mathbb{A}}$ and $\mathbb{H}_{\mathbb{B}}$ are $2^m$ and $2^n$ dimensional vector spaces respectively, over the field of complex numbers $\mathbb{C}$. The dimension of this tensor product space, $\mathbb{H}_{\mathbb{A}} \otimes \mathbb{H}_{\mathbb{B}}$, will be $2^m \times 2^n = 2^{(m+n)} = 2^N$. We have thus transformed the given multipartite pure quantum state, $|\psi\rangle$, into a bipartite pure quantum state to which one can apply the well-known technique of Schmidt decomposition to check whether this state as a bipartite state is entangled or separable.

## 2 Schmidt Decomposition Theorem

Suppose $|\psi\rangle$ is a bipartite pure quantum state of tensor product space, $\mathbb{H}_{\mathbb{A}} \otimes \mathbb{H}_{\mathbb{B}}$, as given above in equation (3) then there exists an orthonormal basis $\{|i_A\rangle\}$ for $\mathbb{H}_{\mathbb{A}}$ and an orthonormal basis $\{|i_B\rangle\}$ for $\mathbb{H}_{\mathbb{B}}$ and nonnegative real numbers, $\{p_i\}$, such that

$$|\psi\rangle = \sum_{i=0}^{2^k-1} p_i |i_A\rangle \otimes |i_B\rangle. \tag{4}$$

where $k = \min\{m, n\}$. These coefficients $p_i$ are called Schmidt coefficients and if the given state, $|\psi\rangle$, is normalized then $\sum_i (p_i)^2 = 1$.

The expansion given above in equation (4) is known as the Schmidt decomposition. The Schmidt number (also called Schmidt rank) is the number of nonzero Schmidt coefficients $p_i$. The Schmidt number is used for the detection of entanglement in bipartite systems. The Schmidt number is 1 for all separable states and for entangled states it is always greater than 1. This is a very important theorem as it acts as an important tool to detect the entanglement in bipartite pure quantum states. Schmidt number characterizes entanglement of any general bipartite systems and not only of 2-qubit systems, i.e. in general $\mathbb{H}_{\mathbb{A}}$, $\mathbb{H}_{\mathbb{B}}$ can be of any arbitrary dimension. To understand Schmidt decomposition in detail let us compare the two representations for the same quantum state $|\psi\rangle$ given in equations (3) and (4). The comparison shows that in equation (3) two different indices, $u$ and $v$, are used on the two sets of basis vectors, $|u\rangle$, $|v\rangle$, to incorporate all the cross-terms, i.e. the maximum number of terms in equation (3) could be $2^m \times 2^n = 2^{(m+n)} = 2^N$ and the expansion coefficients could be complex. But in equation (4) all the expansion coefficients are nonnegative real numbers and all the cross-terms have vanished, and consequently the summation is over a single index, $i$, and further the maximum number of terms in this equation (4) could be $2^k$ where $k = \min\{m, n\}$. Thus Schmidt decomposition provides to an arbitrary bipartite pure quantum state $|\psi\rangle$ in $\mathbb{H}_{\mathbb{A}} \otimes \mathbb{H}_{\mathbb{B}}$ a special basis, $S_A \otimes S_B$, where $S_A$ corresponds to Schmidt basis for the first part and $S_B$ corresponds to Schmidt basis for the second part in the two parts forming the given bipartite state, such that all the coefficients in this expansion are nonnegative real numbers and no cross-terms appear.

The Schmidt decomposition theorem tells us that it is always possible to construct such a special basis to express an arbitrary bipartite pure quantum state in terms of it. In the case of simple examples one can identify the Schmidt basis simply by observation (an easy guess) but it may not be always that simple, so one needs to have a clear prescription of how to obtain Schmidt bases and Schmidt coefficients for an arbitrary bipartite state. For the sake of completeness we describe in brief how it is done. Note that the density matrix for the arbitrary bipartite pure quantum state, $|\psi\rangle$, in the Schmidt basis is $\rho = |\psi\rangle\langle\psi|$ and

$$\begin{aligned} |\psi\rangle\langle\psi| &= \Big(\sum_{i=0}^{2^k-1} p_i |i_A\rangle|i_B\rangle\Big) \otimes \Big(\sum_{j=0}^{2^k-1} p_j \langle j_A|\langle j_B|\Big) \\ &= \sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1} p_i p_j |i_A\rangle|i_B\rangle\langle j_A|\langle j_B| \end{aligned} \tag{5}$$

Now we trace out the system B and obtain reduced density matrix, $\rho^A = \text{Tr}_B(|\psi\rangle\langle\psi|)$ for system A as

$$\rho^A = \sum_{i=0}^{2^k-1} (p_i)^2 |i_A\rangle\langle i_A| \tag{6}$$

Similarly, we trace out the system A and obtain reduced density matrix, $\rho^B = \text{Tr}_A(|\psi\rangle\langle\psi|)$ for system B as

$$\rho^B = \sum_{i=0}^{2^k-1} (p_i)^2 |i_B\rangle\langle i_B| \tag{7}$$

Thus the reduced density operators are diagonal in the Schmidt bases. It can be easily seen that the density operator is diagonalizable since it is a normal operator and from the spectral decomposition theorem it follows that a diagonalizable operator is diagonal in its own eigenbasis. Consequently, the reduced density operator will also be diagonal in its own eigenbasis and we may use that eigenbasis as the Schmidt basis of the subsystem and inserting that in the given state we may find out the Schmidt basis of the other subsystem. Thus, if a composite system is given then by using the standard prescription [6] one can find out Schmidt decomposition as follows:

**Step 1:** Trace out the second subsystem and obtain the reduced density operator for the first subsystem.

**Step 2:** Compute the eigenvectors and the eigenvalues of the reduced density operator obtained in the previous step. The square roots of these eigenvalues are the Schmidt coefficients and the eigenvectors together form the Schmidt basis of the first subsystem.

**Step 3:** Rewrite the given state such that the first subsystem is in the Schmidt basis states obtained in Step 2. This will automatically reveal the Schmidt basis of the other subsystem.

## 3 The Main Result

We now proceed to state and prove the main result which is to be used in the algorithm that we propose here to perform complete factorization of the given $N$-qubit pure quantum state and to provide complete analysis as far as its entanglement status is concerned.

**Theorem:** The $N$-qubit pure quantum state $|\psi\rangle$ given by (1) can be factored as the product, $|\psi_1\rangle \otimes |\psi_2\rangle$, of an $m-$qubit state $|\psi_1\rangle$ and an $n-$qubit state $|\psi_2\rangle$, where $N = m + n$ if and only if when this state is expressed as given in (2), then as a bipartite state as given in (3), and then by executing the Schmidt decomposition protocol on this bipartite state it is finally expressed as given in (4), then this final expression contains only one term.

**Proof:** Suppose the $N$-qubit state $|\psi\rangle$ given by (1) can be factored as the product, $|\psi_1\rangle \otimes |\psi_2\rangle$, of an $m-$qubit state $|\psi_1\rangle$ and an $n-$qubit state $|\psi_2\rangle$ where let

$$|\psi_1\rangle = \sum_{u=0}^{2^m-1} b_{i_u}|i_u\rangle,$$

$$\text{and} \quad |\psi_2\rangle = \sum_{v=0}^{2^n-1} c_{j_v}|j_v\rangle.$$

Using the expression for the state $|\psi\rangle$ given by (2) and for the states $|\psi_1\rangle$ and $|\psi_2\rangle$ as given above in $|\psi\rangle = (|\psi_1\rangle) \otimes (|\psi_2\rangle)$ we get

$$\sum_{u=0}^{2^m-1}\sum_{v=0}^{2^n-1} a_{i_u j_v}|i_u\rangle \otimes |j_v\rangle = \Big(\sum_{u=0}^{2^m-1} b_{i_u}|i_u\rangle\Big) \otimes \Big(\sum_{v=0}^{2^n-1} c_{j_v}|j_v\rangle\Big). \tag{8}$$

We now replace binary sequences $i_u$ and $j_v$ by labels $u$ and $v$ respectively in (8) which converts the $N$-qubit quantum state on the left hand side into a bipartite (2-qudit) state and the right hand side into the tensor product of two 1-partite (1-qudit) states as follows:

$$\sum_{u=0}^{2^m-1}\sum_{v=0}^{2^n-1} a_{uv}|u\rangle \otimes |v\rangle = \Big(\sum_{u=0}^{2^m-1} b_u|u\rangle\Big) \otimes \Big(\sum_{v=0}^{2^n-1} c_v|v\rangle\Big). \tag{9}$$

From (9) it becomes clear that we have now expressed the given state $|\psi\rangle$ as bipartite state on the left hand side in (9) factors as tensor product of two 1-partite states as given on the right hand side of (9). Therefore, $|\psi\rangle$, expressed as bipartite state on the left hand side is product of two 1-partite states and so separable by definition. So, it now follows that by using Schmidt decomposition theorem if we express the state $|\psi\rangle$ given equation (1) first in the form as given in equation (2) and then as a bipartite state as given in equation (3) and then if we apply Schmidt decomposition theorem to this state to obtain it in the form given in equation (4) then such a Schmidt decomposition will contain only one term, since the Schmidt number of a separable state is 1.

Conversely, suppose we start with the $N$-qubit state $|\psi\rangle$ as given in equation (1) we then express this state as given in (2), we then convert it into a bipartite state as given in (3), and then finally by applying Schmidt decomposition protocol we express the state as given in (4), and suppose we find that this final expression contains only one term, i.e. suppose we find that the value of the Schmidt rank for this state $|\psi\rangle$ is equal to 1. In other words, suppose we obtain $|\psi\rangle = |1_A\rangle \otimes |1_B\rangle$, where $|1_A\rangle \in S_A$ and $|1_B\rangle \in S_B$ and where $S_A$ corresponds to Schmidt basis for the first part and $S_B$ corresponds to Schmidt basis for the second part. It is easy to see that $|1_A\rangle \in S_A$ will be a 1-qudit vector in the $2^m$ dimensional space and $|1_B\rangle \in S_B$ will be a 1-qudit vector in the $2^n$ dimensional space. So, we can express $|\psi\rangle$ as

$$|\psi\rangle = |1_A\rangle \otimes |1_B\rangle = \Big(\sum_{u=0}^{2^m-1} b_u|u\rangle\Big) \otimes \Big(\sum_{v=0}^{2^n-1} c_v|v\rangle\Big). \tag{10}$$

We now do exactly opposite of what was done above, i.e. we express the wave-function $|\psi\rangle$ on the left hand side of (10) in its original form as $N$-qubit state. We then replace the labels $u$ and $v$ on the right hand side of (10) by their corresponding binary sequences $i_u$ and $j_v$ of lengths $m$ and $n$ respectively which thus converts the bipartite (2-qudit) quantum state on the left hand side into a $N$-qubit quantum state and on the right hand side we will now have the tensor product of an $m$-qubit quantum state and an $n$-qubit quantum state. We thus will have the following equation:

$$|\psi\rangle = \Big(\sum_{u=0}^{2^m-1} b_{i_u}|i_u\rangle\Big) \otimes \Big(\sum_{v=0}^{2^n-1} c_{i_v}|i_v\rangle\Big). \tag{11}$$

Thus, the originally given $N$-qubit pure quantum state $|\psi\rangle$ has factors. It is a tensor product of an $m$-qubit pure quantum state and an $n$-qubit pure quantum state as desired.

## 4 The Algorithm

We now proceed to present our algorithm, based on the above theorem, for complete factorization of an arbitrary $N$-qubit pure quantum state. This algorithm provides a systematic procedure for complete analysis of the nature of $N$-qubit pure quantum states. Our algorithm will completely specify the following: Whether the given $N$-partite pure quantum state is completely separable ($N$ factors), or completely entangled (no factors), or partially entangled having entangled factors of different sizes which cannot be factored further. The steps of the algorithm are as follows:

**(i)** We have the given $N$-qubit pure state $|\psi\rangle$ in terms of the computational basis as

$$|\psi\rangle = \sum_{s=0}^{2^N-1} a_{r_s}|r_s\rangle\,, \tag{12}$$

where the basis vectors $|r_s\rangle$ are ordered lexicographically.

**(ii)** Now our aim is to check as first step (using the Theorem just proved above) whether given $|\psi\rangle$ has a linear (1-qubit) factor and an $(N-1)$-qubit factor, i.e. the case with $m=1$ and $n=N-1$ of the above proved theorem. To check this we rewrite the above state, $|\psi\rangle$, as

$$|\psi\rangle = \sum_{u=0}^{1} |i_u\rangle \otimes \Big[\sum_{v=0}^{2^{(N-1)}-1} a_{i_u j_v}|j_v\rangle\Big]. \tag{13}$$

Here the basis vectors ordered lexicographically are $|i_0\rangle = |0\rangle, |i_1\rangle = |1\rangle$ (each of length 1) and $|j_0\rangle = |00\cdots 0\rangle, \ldots, |j_{2^{(N-1)}-1}\rangle = |11\cdots 1\rangle$ (each of length $N-1$). We convert the above state in (13) into a bipartite state. To achieve this we replace the binary sequences by the corresponding numbers which leads to replacement of the corresponding basis vectors as follows: $|i_0\rangle$ becomes $|0\rangle$ and $|i_1\rangle = |1\rangle$. Similarly, $|j_0\rangle, \ldots, |j_{2^{(N-1)}-1}\rangle$ (each of length $N-1$) becomes $|0\rangle, |1\rangle, |2\rangle, \ldots, |2^{(N-1)-1}\rangle$ (each of length 1). Thus, the above quantum state $|\psi\rangle$ in (13) becomes

$$|\psi\rangle = \sum_{u=0}^{1} |u\rangle \otimes \Big[\sum_{v=0}^{2^{(N-1)}-1} a_{uv}|v\rangle\Big]. \tag{14}$$

We have thus converted the state in (13) into a bipartite state in (14). We now use the above theorem and apply Schmidt decomposition procedure to this bipartite state and obtain the Schmidt rank.

**(iii)** Now there are two cases.

***Case I***

If Schmidt rank = 1, then by above theorem we will obtain

$$|\psi\rangle = |1_A\rangle \otimes |1_B\rangle = \Big(\sum_{u=0}^{1} b_u|u\rangle\Big) \otimes \Big(\sum_{v=0}^{2^{(N-1)}-1} c_v|v\rangle\Big). \tag{15}$$

Now by replacing the left hand side by original $N$-qubit state and on the the right hand side by replacing the numbers $u, v$ by their corresponding binary sequences $i_u, j_v$ in (15) we get

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle,$$
$$\text{where} \quad |\psi_1\rangle = \sum_{u=0}^{1} b_{i_u}|i_u\rangle \quad \text{and} \quad |\psi_2\rangle = \sum_{v=0}^{2^{(N-1)}-1} c_{j_v}|j_v\rangle,$$

where $i_u$ and $j_v$ are the binary sequences of lengths 1 and $(N-1)$, respectively, corresponding to numbers $u, v$. In this case we go back to step (i) with $|\psi\rangle = |\psi_2\rangle$.

***Case II***

If Schmidt rank $\neq 1$, then as per above theorem we do not get a factor, $|\psi_1\rangle$, as above to $|\psi\rangle$, namely, $|\psi_1\rangle = b_{i_0}|i_0\rangle + b_{i_1}|i_1\rangle$. In this case our aim is to check as the next step whether given $|\psi\rangle$ can be expressed as product of a 2-qubit factor and an $(N-2)$-qubit factor i.e. the case with $m = 2$ and $n = (N-2)$ of the above proved theorem. For this we proceed with the originally given state $|\psi\rangle$ as given in the next step (iv).

**(iv)** To check this we rewrite the above state, $|\psi\rangle$, as

$$|\psi\rangle = \sum_{u=0}^{2^2-1} |i_u\rangle \otimes \Big[\sum_{v=0}^{2^{(N-2)}-1} a_{i_u j_v}|j_v\rangle\Big]. \tag{16}$$

Here the basis vectors ordered lexicographically are $|i_0\rangle = |00\rangle$, $|i_1\rangle = |01\rangle$, $|i_2\rangle = |10\rangle$, $|i_3\rangle = |11\rangle$ (each of length 2) and $|j_0\rangle = |00\cdots 0\rangle, \ldots, |j_{2^{(N-2)}-1}\rangle = |11\cdots 1\rangle$ (each of length $N-2$). We convert the above state into a bipartite state. To achieve this we replace the binary sequences in the above state in (16) by the corresponding numbers which leads to replacement of the corresponding basis vectors as follows: $|i_0\rangle = |0\rangle$, $|i_1\rangle = |1\rangle$, $|i_2\rangle = |2\rangle$, $|i_3\rangle = |3\rangle$ (each of length 1) Similarly, $|j_0\rangle, \ldots, |j_{2^{(N-2)}-1}\rangle$ (each of length $N-2$) becomes $|0\rangle, \ldots, |2^{(N-2)}-1\rangle$ (each of length 1). Thus, the above quantum state $|\psi\rangle$ in (16) becomes

$$|\psi\rangle = \sum_{u=0}^{2^2-1} |u\rangle \otimes \Big[ \sum_{v=0}^{2^{(N-2)}-1} a_{uv}|v\rangle \Big]. \tag{17}$$

We now use the above theorem and apply Schmidt decomposition procedure to this bipartite state and obtain the Schmidt rank. Again there will be two cases.

***Case I***

If Schmidt rank = 1, then by above theorem we will obtain

$$|\psi\rangle = |1_A\rangle \otimes |1_B\rangle = \Big( \sum_{u=0}^{2^2-1} b_u|u\rangle \Big) \otimes \Big( \sum_{v=0}^{2^{(N-2)}-1} c_v|v\rangle \Big). \tag{18}$$

Now, by replacing back the numbers $u, v$ by their corresponding binary sequences $i_u, j_v$ in (18) we get

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle,$$
$$\text{where} \quad |\psi_1\rangle = \sum_{u=0}^{2^2-1} b_{i_u}|i_u\rangle \quad \text{and} \quad |\psi_2\rangle = \sum_{v=0}^{2^{(N-2)}-1} c_{j_v}|j_v\rangle,$$

where $i_u$ and $j_v$ are the binary sequences of lengths 2 and $(N-2)$, respectively, replaced for the numbers $u, v$. In this case we go back to step (i) with $|\psi\rangle = |\psi_2\rangle$.

***Case II***

If Schmidt rank $\neq 1$, then as per above theorem we do not get a factor, $|\psi_1\rangle$, to $|\psi\rangle$, like, $|\psi_1\rangle = b_{i_0}|i_0\rangle + b_{i_1}|i_1\rangle + b_{i_2}|i_2\rangle + b_{i_3}|i_3\rangle$ In this case our aim is to check as the next step whether given $|\psi\rangle$ has a 3-qubit factor and an $(N-3)$-qubit factor i.e. the case with $m = 3$ and $n = (N-3)$ of the above proved theorem. For this we proceed with the originally given state $|\psi\rangle$ as given in the next step (v).

**(v)** To check this we rewrite the above state, $|\psi\rangle$, as

$$|\psi\rangle = \sum_{u=0}^{2^3-1} |i_u\rangle \otimes \Big[ \sum_{v=0}^{2^{(N-3)}-1} a_{i_u j_v}|j_v\rangle \Big]. \tag{19}$$

Here the basis vectors ordered lexicographically are $|i_0\rangle = |000\rangle, \ldots, |i_7\rangle = |111\rangle$ (each of length 3) and $|j_0\rangle = |00\cdots0\rangle, \ldots, |j_{2^{(N-3)}-1}\rangle = |11\cdots1\rangle$ (each of length $N-3$). In order to convert the above state into a bipartite state we replace the binary sequences in the above state by the numbers which leads to replacement of the corresponding basis vectors as follows: $|i_0\rangle = |0\rangle, |i_1\rangle = |1\rangle,$

$|i_2\rangle = |2\rangle, \cdots, |i_7\rangle = |7\rangle$ (each of length 1) Similarly, $|j_0\rangle, \ldots, |j_{2^{(N-3)}-1}\rangle$ (each of length $N-3$) becomes $|0\rangle, \ldots, |2^{(N-3)}-1\rangle$ (each of length 1). Thus, the above quantum state $|\psi\rangle$ in (19) becomes

$$|\psi\rangle = \sum_{u=0}^{2^3-1} |u\rangle \otimes \Big[ \sum_{v=0}^{2^{(N-3)}-1} a_{uv}|v\rangle \Big]. \tag{20}$$

We now use the above theorem and apply Schmidt decomposition procedure to this bipartite state and obtain the Schmidt rank. Again there will be two cases.

***Case I***

If Schmidt rank = 1, then by above theorem we will obtain

$$|\psi\rangle = |1_A\rangle \otimes |1_B\rangle = \Big( \sum_{u=0}^{2^3-1} b_u|u\rangle \Big) \otimes \Big( \sum_{v=0}^{2^{(N-3)}-1} c_v|v\rangle \Big). \tag{21}$$

By replacing back the numbers $u, v$ by their corresponding binary sequences $i_u, j_v$ in (21) we get

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle,$$
$$\text{where} \quad |\psi_1\rangle = \sum_{u=0}^{2^3-1} b_{i_u}|i_u\rangle \quad \text{and} \quad |\psi_2\rangle = \sum_{v=0}^{2^{(N-3)}-1} c_{j_v}|j_v\rangle,$$

where $i_u$ and $j_v$ are the binary sequences of lengths 3 and $(N-3)$, respectively, for numbers $u, v$. In this case we go back to step (i) with $|\psi\rangle = |\psi_2\rangle$.

***Case II***

If Schmidt rank $\neq 1$, then as per above theorem we do not get a factor to $|\psi\rangle$ like $|\psi_1\rangle = b_{i_0}|i_0\rangle + b_{i_1}|i_1\rangle + b_{i_2}|i_2\rangle + \cdots, +b_{i_7}|i_7\rangle$. We then proceed to check as the next step whether given $|\psi\rangle$ has a 4-qubit factor and an $(N-4)$-qubit factor i.e. the case with $m = 4$ and $n = (N-4)$ of the above proved theorem, and so on. Thus, as above the algorithm continues until $|\psi\rangle$ is completely factored.

## 5 The Computational Complexity

We now proceed to obtain the computational complexity of the above algorithm. In this algorithm we have extended the idea of applying Schmidt decomposition procedure to check the entanglement status of a multipartite pure quantum state and have developed an easy algorithm to perform complete analysis of the given $N-$qubit pure quantum state. We successfully manage the full factorization of the given multipartite pure quantum state and specify the following: Whether the given an $N-$qubit pure quantum state is completely separable ($N$ factors), or completely entangled (no factors), or partially entangled having entangled factors of different sizes which being entangled cannot be factored further. The complexity of the above algorithm clearly depends upon two things: Firstly, on the complexity of Schmidt decomposition procedure which is used in this algorithm and secondly, on how many time we need to use this Schmidt decomposition procedure in this algorithm. Now,

**(i)** To check whether the given $N-$qubit state under consideration has as an $m-$qubit factor and an $n-$qubit factor, $m+n=N$, we need to express this state as a bipartite state and then by applying Schmidt decomposition protocol we have to express this state as given in the equation (4) above, and find whether this expression as given in the equation (4) contains only one term, i.e. we need to find out whether the value of the Schmidt rank for this state is equal to 1.

**(ii)** Also, an $N-$qubit state can have at most $N$ factors (when this multipartite state under consideration is separable and therefore has $N$ 1-qubit factors), therefore, we will need to perform Schmidt decomposition at most $N$ times.

The so called Schmidt decomposition is nothing but the singular value decomposition (SVD) in disguise. Schmidt decomposition is essentially a restatement of the SVD in a different context. Therefore, to calculate the Schmidt decomposition one can proceed with rewriting this state as a matrix and then obtain the SVD (page 433, [7]). Thus, the computational effort to find Schmidt decomposition is identical with the computational effort required to find the SVD. Now, the algorithm called the R-SVD algorithm to obtain the SVD of a $P \times Q$ matrix requires steps that are of the order, $O(kP^2Q + k'Q^3)$, where $k$ and $k'$ are constants, which are 4 and 22 respectively for this version of the SVD algorithm called the R-SVD algorithm (page 517, [8]).

Clearly, to check (i), i.e. to check whether an $N-$qubit state has as an $m-$qubit factor and an $n-$qubit factor we will need to obtain the SVD of the matrix of size $P \times Q$ for which $P = 2^m$ and $Q = 2^n$. Therefore, the complexity of this task becomes of the order of $O(k(2^m)^2 2^n + k'(2^n)^3)$ by appropriately substituting for $P$ and $Q$ in the above given formula for the complexity of this so called the R-SVD algorithm. Now, as per (ii) the $N-$qubit state can have at most $N$ factors, so we will need to obtain the SVD of the various matrices, for certain values of $P$ and $Q$, at most $N$ times. Thus, the total overall complexity of our algorithm discussed above will be at most of the order equal to $N$ times the maximum of the above complexity over $m$, where $m \in \{1, 2, \cdots, 2^N - 1\}$ for testing whether an $N-$qubit state has as an $m-$qubit factor and an $n-$qubit factor, where $m + n = N$, i.e. the total overall complexity for our algorithm discussed above will be at most of the order of

$$O(N \times \max_m \{(k(2^m)^2 2^n + k'(2^n)^3))\},$$

where $k$ and $k'$ are constants equal to 4 and 22 respectively if one selects to use the version of the SVD algorithm called the R-SVD algorithm, [8].

## 6 Examples

We solve two examples using our new algorithm developed in section 4 above. In the first example we consider a separable 3-qubit state and in the second example we consider an entangled 3-qubit state with one linear factor and one 2-qubit entangled factor.

**(i)** A separable 3-qubit state: Suppose

$$|\psi\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |001\rangle - |010\rangle + |011\rangle + |100\rangle - |101\rangle - |110\rangle + |111\rangle).$$

To check whether $|\psi\rangle$ has a linear factor (on left) as per the above discussed algorithm, we rewrite $|\psi\rangle$ as a bipartite state thus:

$$|\psi\rangle = \frac{1}{\sqrt{8}}(|00\rangle - |01\rangle - |02\rangle + |03\rangle + |10\rangle - |11\rangle - |12\rangle + |13\rangle).$$

Therefore,

$$\begin{aligned}\rho^A &= \mathrm{Tr}_B(|\psi\rangle\langle\psi|)\\ &= {}_B\langle 0|\psi\rangle\langle\psi|0\rangle_B + {}_B\langle 1|\psi\rangle\langle\psi|1\rangle_B + {}_B\langle 2|\psi\rangle\langle\psi|2\rangle_B + {}_B\langle 3|\psi\rangle\langle\psi|3\rangle_B\\ &= \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|).\end{aligned}$$

Therefore,

$$\rho^A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

We get the eigenvalues for the matrix $\rho^A$ as 1 and 0. Therefore, the Schmidt number (number of nonzero eigenvalues) is 1 indicating that the state is separable and we get the factored state as per the steps of the algorithm as

$$\begin{aligned}|\psi\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)\\ &= |\psi_1\rangle \otimes |\psi_2\rangle.\end{aligned}$$

So, as per the algorithm, we now take $|\psi_2\rangle$ as $|\psi\rangle$ and proceed again from beginning and follow the steps of the algorithm. By this way we finally get the complete factorization of originally given state as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Hence, this state $|\psi\rangle$ has three linear factors and thus this state is separable.

**(ii)** An entangled 3-qubit state: Suppose

$$|\psi\rangle = \frac{1}{\sqrt{6}}(|000\rangle + |001\rangle + |010\rangle + |100\rangle + |101\rangle + |110\rangle).$$

To check whether $|\psi\rangle$ has a linear factor (on left) as per the above discussed algorithm, we rewrite $|\psi\rangle$ as a bipartite state thus:

$$|\psi\rangle = \frac{1}{\sqrt{6}}(|00\rangle + |01\rangle + |02\rangle + |10\rangle + |11\rangle + |12\rangle).$$

Therefore,

$$\begin{aligned}\rho^A &= \mathrm{Tr}_B(|\psi\rangle\langle\psi|)\\ &= {}_B\langle 0|\psi\rangle\langle\psi|0\rangle_B + {}_B\langle 1|\psi\rangle\langle\psi|1\rangle_B + {}_B\langle 2|\psi\rangle\langle\psi|2\rangle_B + {}_B\langle 3|\psi\rangle\langle\psi|3\rangle_B\\ &= \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|).\end{aligned}$$

Therefore,

$$\rho^A = \begin{bmatrix} 1/2 & 12 \\ 12 & 12 \end{bmatrix}.$$

We get the eigenvalues for the matrix $\rho^A$ as 1 and 0. Therefore the Schmidt number (number of nonzero eigenvalues) is 1 indicating that the state is separable and we get the factored state as per the steps of the algorithm as

$$\begin{aligned}|\psi\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)\\ &= |\psi_1\rangle \otimes |\psi_2\rangle.\end{aligned}$$

So, as per the algorithm, we now take $|\psi_2\rangle$ as $|\psi\rangle$ and proceed again from beginning and follow the steps of the algorithm. Thus, We now consider the 2-qubit state obtained above:

$$|\psi_2\rangle = |\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle),$$

$$\begin{aligned}\rho^A &= \mathrm{Tr}_B(|\psi\rangle\langle\psi|)\\ &= {}_B\langle 0|\psi\rangle\langle\psi|0\rangle_B + {}_B\langle 1|\psi\rangle\langle\psi|1\rangle_B\\ &= \frac{1}{3}(2|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|).\end{aligned}$$

Thus,

$$\rho^A = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix}.$$

The eigenvalues of $\rho^A$ are $\frac{1}{2}+\frac{\sqrt{5}}{6}$ = 0.873 and $\frac{1}{2}-\frac{\sqrt{5}}{6}$ = 0.127. Therefore, Schmidt rank is 2 and Schmidt coefficients are $\sqrt{(0.873)}$ = 0.934 and $\sqrt{(0.127)}$ = 0.356, with eigenvectors

$$|u_1\rangle_A = \begin{bmatrix} \dfrac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

and

$$|u_2\rangle_A = \begin{bmatrix} \dfrac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix},$$

respectively. Normalizing these two eigenvectors we obtain the Schmidt basis for the subsystem A as

$$|u_1^n\rangle_A = \begin{bmatrix} 0.851 \\ 0.526 \end{bmatrix}$$

and

$$|u_2^n\rangle_A = \begin{bmatrix} -0.526 \\ 0.851 \end{bmatrix}.$$

Thus there are two nonzero Schmidt coefficients and therefore Schmidt number is 2 indicating that this state is entangled. Thus, this 3-qubit state has one linear (1-qubit) factor and one 2-qubit entangled factor.

## 7 Conclusions

We have developed here an algorithm to completely determine the entanglement status of multipartite pure quantum states. It will be a very useful tool to perform complete analysis and to determine all the factors (separable as well as entangled factors) of multipartite pure quantum states.

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