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The Role of Acceleration in the Twin Paradox

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Abstract. We exemplify the role of acceleration in the twin paradox by calculating explicitly the elapsed proper time of each twin, according to each twin. In the limit that the coasting periods are much longer than the accelerating periods for the travelling twin, we find that the accelerating portions of the trajectory is crucial for the calculation of the elapsed proper time of the inertial twin according to the travelling twin, but irrelevant for all other calculations. We point out that the naive symmetry between the two twins is actually explicitly broken as soon as the turn around point is specified in the inertial twins reference frame.

KEY WORDS: special relativity, twin paradox.

1 Introduction

In the twin paradox a set of twins, A and B, synchronize their watches in an inertial reference frame and then one twin (B) takes a journey to a distant object, defined in the inertial reference frame, and then returns. They compare their watches once they are together again and they find that the twin that travelled is younger. This is perfectly natural from the point of view of special relativity since it is understood that time dilates for an observer that moves relative to one that does not. The situation is paradoxical since it is in fact relative to decide or determine which twin has moved. Each twin thinks the other is the one that has moved. However, the situation is actually not exactly symmetric. One twin experiences acceleration while the other does not. Furthermore, the turn around point is defined in the inertial reference frame, which is in fact the reference frame of the twin that does not move (A) and not the (non-inertial) reference frame of the the twin that does move and does feel acceleration (B). These differences break the symmetry between the twins and lead to a resolution of the paradox.

2 Limit of Short Accelerating Periods

The space-time diagrams of the two twins is given in Figure 1.

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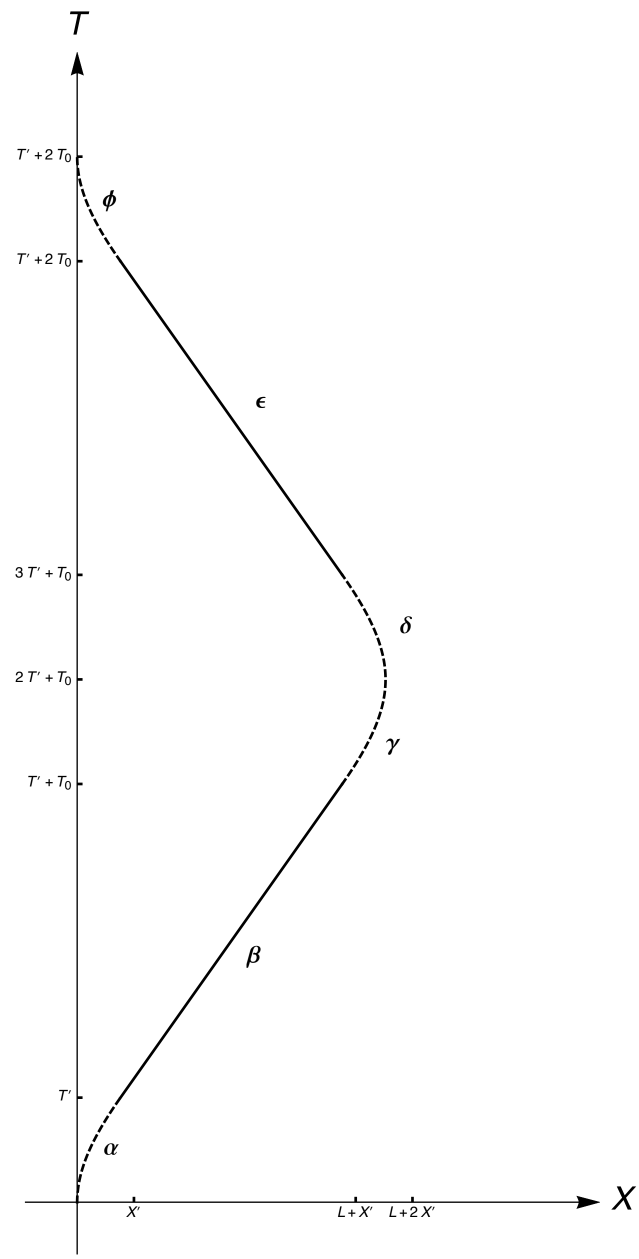

Figure 1. Worldline of B, illustrated in reference frame R

We consider the limit that the accelerating periods are short compared to the coasting periods. Suppose this is the case for the trajectories shown in Figure 1, $T' \ll T_0$. Using the notation that, for example, $\Delta\tau_{\rm A}(\rm B)$ is the elapsed proper time of B according to A, we must be able to show that

$$\Delta\tau_{\rm A}(\rm A) = \Delta\tau_{\rm B}(\rm A) \tag{1}$$

and

$$\Delta\tau_{\rm A}(\rm B) = \Delta\tau_{\rm B}(\rm B). \tag{2}$$

Neglecting the accelerating periods (calling the acceleration g), it is an easy exercise in special relativity to calculate that during the inertial parts of the trajectories (according to each twin respectively) we have

$$\Delta\tau_{\rm A}(\rm A)|_{\text{inertial}} = 2T_0 \tag{3}$$

$$\Delta\tau_{\rm B}(\rm A)|_{\text{inertial}} = 2T_0/\gamma^2 \tag{4}$$

while

$$\Delta\tau_{\rm A}(\rm B)|_{\text{inertial}} = 2T_0/\gamma \tag{5}$$

$$\Delta\tau_{\rm B}(\rm B)|_{\text{inertial}} = 2T_0/\gamma \tag{6}$$

where $\gamma = 1/\sqrt{1 - V^2}$ the usual relativistic factor.

The equality of $\Delta\tau_{\rm A}(\rm B)|_{\text{inertial}} = 2T_0/\gamma = \Delta\tau_{\rm B}(\rm B)|_{\text{inertial}}$ is gratifying however the lack of equality $\Delta\tau_{\rm A}(\rm A)|_{\text{inertial}} = 2T_0 \neq 2T_0\gamma^2 = \Delta\tau_{\rm B}(\rm B)|_{\text{inertial}}$ is discomfiting, and has to be corrected by taking into account the elapsed proper time of A according to B during the accelerating period of B (which has of course been neglected in obtaining Eqn. (4)). The contribution of the accelerating periods is of order $o(1/g)$ for Eqns. (3), (5) and (6) but it is of $o(1)$ for Eqn. (4). Calculating the correction to Eqn. (4) from the accelerating phase is the necessary exercise that demonstrates that the twin paradox is in fact not a paradox.

In what follows, we will only show in detail the calculation of Eqn. (4) as the others are straightforward. We will see how the calculation of the elapsed proper time of A according to B also gives $\Delta\tau_{\rm B}(\rm A) = 2T_0$.

3 Accelerating Phase

The coordinate system of B, during the accelerating phase cannot be an inertial reference frame. We consider the transformation of coordinates between (T, X) for twin A and $(\tilde{T}, \tilde{X})$ for twin B

$$rClX = \left(\frac{1}{g} + \tilde{X}\right)\cosh(g\tilde{T}) - \frac{1}{g} \tag{7}$$

$$T = \left(\frac{1}{g} + \tilde{X}\right)\sinh(g\tilde{T}) \tag{8}$$

which correspond to the hyperbolas

$$\left(X + \frac{1}{g}\right)^2 - T^2 = \left(\frac{1}{g} + \tilde{X}\right)^2. \tag{9}$$

The trajectory of twin B is at $\tilde{X} = 0$ while that of twin A is at $X = 0$. The coordinates, $(\tilde{T}, \tilde{X})$, are called Kottler-Rindler [1–3] coordinates.

The position of twin A, $X = 0$, corresponds to

$$\tilde{X}(\tilde{T}) = \frac{1}{g}\left(\operatorname{sech}(g\tilde{T}) - 1\right) \tag{10}$$

which asymptotes to $\tilde{X} = -\frac{1}{g}$. Notice that $\operatorname{sech}(x) \leq 1$, hence twin B sees twin A as moving in the negative $\tilde{X}$ direction, but as far as twin B is concerned, twin A never manages to move past $-\frac{1}{g}$ as $\tilde{T} \to \infty$. This boundary is a coordinate artefact known as the Rindler horizon or edge of the Rindler wedge. For short enough accelerations, the Kottler-Rindler coordinates are perfectly fine.

The metric in Kottler-Rindler coordinates is obtained from the Minkowski metric and the differentials

$$dT = d\tilde{X}\sinh(g\tilde{T}) + \left(\frac{1}{g} + \tilde{X}\right) g\cosh(g\tilde{T}) \tag{11}$$

$$dX = d\tilde{X}\cosh(g\tilde{T}) + \left(\frac{1}{g} + \tilde{X}\right) g\sinh(g\tilde{T}). \tag{12}$$

Then the metric is given by

$$\begin{aligned} d\tau^2 &= dT^2 - dX^2 \\ &= \left(1 + g\tilde{X}\right)^2 d\tilde{T}^2 - d\tilde{X}^2. \end{aligned} \tag{13}$$

Thus for twin B at $\tilde{X} = 0$, we have $d\tau^2 = d\tilde{T}^2$ and then clearly $\Delta\tau_{\rm B}({\rm B}) = \Delta\tilde{T}_{\rm B}$.

The periods of deceleration, γ and δ, will be associated with different hyperbolas. First we replace $g \to -g$ in Eqns. (7) and (8). This gives

$$rClX = \left(\tilde{X} - \frac{1}{g}\right)\cosh(g\tilde{T}) + \frac{1}{g} \tag{14}$$

$$T = -\left(\tilde{X} - \frac{1}{g}\right)\sinh(g\tilde{T}). \tag{15}$$

with corresponding hyperbola

$$\left(X - \frac{1}{g}\right)^2 - T^2 = \left(\tilde{X} - \frac{1}{g}\right)^2. \tag{16}$$

We are interested in the left branch of this set of hyperbolas, which is the decelerating branch. This requires that $\tilde{X} \le \frac{1}{g}$ which then imposes that $X \le \frac{1}{g}$. Because of this, the time $\tilde{T}$ and T run in the same direction, in Eqn. (15), $-\left(\tilde{X} - \frac{1}{g}\right)$ is positive.

We want that the trajectory of B during the decelerating phase to pass through the turn around point when $\tilde{T} = \tilde{T}_D$ (the value of $\tilde{T}_D$ is actually not required for our analysis, it can of course be determined). The turn around point is defined in the coordinates of A as $(T_D, X_D) = (T_0 + 2T', X_0 + 2X')$. Thus we shift the hyperbola as

$$rClX - X_D = \left(\tilde{X} - \frac{1}{g}\right)\cosh(g\left(\tilde{T} - \tilde{T}_D\right)) + \frac{1}{g} \tag{17}$$

$$T - T_D = -\left(\tilde{X} - \frac{1}{g}\right)\sinh(g\left(\tilde{T} - \tilde{T}_D\right)). \tag{18}$$

Clearly

$$\left(X - X_D - \frac{1}{g}\right)^2 - (T - T_D)^2 = \left(\tilde{X} - \frac{1}{g}\right)^2 \tag{19}$$

which is the equation of a family of hyperbolas, parametrized by $\tilde{X}$, symmetric about $\left(T_D, X_D + \frac{1}{g}\right)$. We choose the left branch (decelerating) by imposing that $\tilde{X} \le \frac{1}{g}$. As mentioned above, the hyperbola for $\tilde{X} = 0$ corresponds to the trajectory of twin B and at $\tilde{T} = \tilde{T}_D$, twin B will be at the turn around point (X_D, T_D) in the coordinate system of twin A. The metric in these coordinates is exactly as before except $g \to -g$

$$d\tau^2 = \left(1 - g\tilde{X}\right)^2 d\tilde{T}^2 - d\tilde{X}^2. \tag{20}$$

We will proceed to compute the elapsed proper time of A according to B during the decelerating phase. It is the decelerating phase that makes up for all the time lost that seems to be making A younger than B, except for correction $o(1/g)$, and reverses the relative ageing. The metric in Eqn. (13) changes with $g \to -g$,

$$d\tau = \left(\left(1 - g\tilde{X}\right)^2 - \left(\frac{d\tilde{X}}{d\tilde{T}}\right)^2\right)^{1/2} d\tilde{T} \tag{21}$$

and replacing $X = 0$ in Eqn. (17) yields $\tilde{X}_{\rm A}$

$$\tilde{X}_{\rm A} = -\left(X_D + \frac{1}{g}\right)\operatorname{sech}\left(g(\tilde{T} - \tilde{T}_D)\right) + \frac{1}{g}. \tag{22}$$

Then we get

$$d\tau_{{\rm A}\gamma}({\rm B}) = \left((1 + gX_D)\operatorname{sech}^2\left(g(\tilde{T}_{\rm A} - \tilde{T}_D)\right)\right) d\tilde{T}_{\rm A} \tag{23}$$

and integrating from $\tilde{T}_D - \tilde{T}'$ to $\tilde{T}_D$ gives

$$\begin{aligned}\Delta\tau_{A\gamma}(\mathrm{B}) &= (1+gX_D)\,\frac{1}{g}\tanh\left(g\tilde{T}'\right)\\ &= (1+g(X_0+2X'))\,\frac{T'}{(1+(gT')^2)^{1/2}}.\end{aligned} \tag{24}$$

We note that $\Delta\tau_{A\gamma}(\mathrm{B})$ can be as large because of the additional term

$$g(X_0+2X')\frac{T'}{(1+(gT')^2)^{1/2}}. \tag{25}$$

We will not show it explicitly here, but it is easy to see that $\Delta\tau_{A\alpha}(\mathrm{B}) \sim o(1/g)$, thus negligible for large g. The limit large g is taken simultaneously as $T' \to 0$ so that gT' remains constant. This means that the velocity at the end of the acceleration is independent of how large the acceleration is,

$$V = \frac{gT'}{(1+(gT')^2)^{1/2}} \tag{26}$$

and $T' \sim 0(1/g)$. Then

$$\begin{aligned}\Delta\tau_{A}(\mathrm{B}) &= (\Delta\tau_{A\alpha}(\mathrm{B}) + \Delta\tau_{A\beta}(\mathrm{B}) + \Delta\tau_{A\gamma}(\mathrm{B}))\times 2\\ &= 2\left(\Delta\tau_{A\beta}(\mathrm{B}) + \Delta\tau_{A\gamma}(\mathrm{B}) + o(1/g)\right)\\ &= \frac{2T_0}{\gamma_V^2} + (1+g(X_0+2X'))\,\frac{2T'}{(1+(gT')^2)^{1/2}} + o(1/g)\\ &= \frac{2T_0}{\gamma_V^2} + ((X_0+2X'))\,\frac{2gT'}{(1+(gT')^2)^{1/2}} + o(1/g)\end{aligned} \tag{27}$$

Using the expression for the velocity above and

$$X' = \frac{1}{g}\left(\left(1+(gT')^2\right)^{1/2} - 1\right) \tag{28}$$

and $X_0 = VT_0$ we get

$$\begin{aligned}\Delta\tau_{A}(\mathrm{B}) &= \left(1-V^2\right)2T_0 + 2V^2T_0\\ &\quad + 4\frac{V}{g}\left(\left(1+(gT')^2\right)^{1/2} - 1\right) + o(1/g)\\ &= 2T_0 + 4\left(1+(gT')^2\right)^{1/2}\frac{T'}{(1+(gT')^2)^{1/2}} + o(1/g)\\ &= 2T_0.\end{aligned} \tag{29}$$

In the last equality, we have dropped the terms $o(1/g)$ including T'. Actually a more careful analysis shows that all terms exactly add up to $4T'$ [4]. Therefore we reproduce that the elapsed proper time of A according to B is

$$\Delta\tau_{\rm A}({\rm B}) = 2T_0 + 4T' = \Delta\tau_{\rm A}({\rm A}), \tag{30}$$

and of course

$$\Delta\tau_{\rm B}({\rm A}) = \Delta\tau_{\rm B}({\rm B}) < \Delta\tau_{\rm A}({\rm B}) = \Delta\tau_{\rm A}({\rm A}). \tag{31}$$

Thus the twin paradox is completely resolved, the travelling twin B is younger than the inertial twin A after the journey, and we have explicitly shown how to do the calculation of the elapsed proper time by each twin for of each twin.

4 Conclusion

A complete resolution of the twin paradox requires the calculation of the elapsed proper time of each twin from the point of view of both twins. The accelerating period during the turnaround of twin B is crucial for the calculation according to twin B of the elapsed proper time of twin A. It was observed that already for just the inertial parts of the trajectories, the situation is asymmetric with respect to the two twins. The turnaround point is defined *ab initio* in the inertial frame of twin A. A observes B moving a certain distance to the turnaround point and back. On the other hand, during the inertial parts of B's trajectory, he or she observes A moving a shorter distance back and forth, this distance being shorter because of the standard Lorentz contraction. The elapsed proper time during these asymmetrical, inertial parts of the trajectory as calculated by the travelling twin B for the inertial twin A is surprisingly shorter than the same calculation by B of his or her elapsed proper time during the same parts of the trajectory. The elapsed proper time of A calculated by B during the decelerating part of his or her trajectory is crucial and makes up for the missing proper time.

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