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Helicity and the Dissipation of Energy in Incompressible Fluids

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Abstract. This is mostly a pedagogical review of some mathematical aspects of incompressible fluid flow. Many Lie theoretic ideas familiar to particle physicists illuminate aspects of fluid mechanics. There is a close analogy with the dynamics of a rigid body. For example, helicity is a quadratic Casimir invariant analogous to the square of angular momentum. Making another analogy with the Bogomolny inequality in the theory of instantons, I derive a new inequality for the dissipation of energy, relating it to helicity.

KEY WORDS: helicity, fluid, Navier-Stokes equations

1 Lightning Review of Fluid Mechanics

Next to Celestial Mechanics, Fluid Mechanics is the oldest part of physics. Yet, some of the deepest unsolved problems in physics are in this area; the most important being turbulence. Much progress has been made in celestial mechanics (or the theory of dynamical systems, as it has been rebranded) inspired by Poincare's work in topology, and from the use of numerical computations. Fluid Mechanics too, has been revolutionized by numerical computations. More inspiring to a theorist are the topological ideas of Arnold's school [1], which could complement the statistical theories of Kolmogorov [2] and Onsager [3]. Since scale invariance is central to any approach to turbulence, it is reasonable to hope that Wilson's renormalization group will lead to a systematic theory of turbulence.

My aim in this paper is quite modest: to explain the basics of fluid mechanics in a language that ought to be familiar to particle/condensed matter physicists. And to present a result obtained by analogy with the well known Bogomolny inequality of instanton theory [4].

I will confine myself to incompressible fluid flow. (For a more detailed discussion, see [5].)This means that density is a constant: the fluid speed is small compared to the speed of sound. In this limit conservation of mass reduces to the condition of incompressibility

$$\text{div } v = 0\,.$$

Newton's laws applied to a small cube transported by the fluid gives the equations of motion

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p + \nu \nabla^2 v, \quad \text{Navier} - \text{Stokes Equations} \tag{1}$$

Here, p is the pressure divided by density and ν the kinematical viscosity (which has units of $\frac{\text{length}^2}{\text{time}}$). The l.h.s. is the acceleration of a fluid element and the r.h.s. is the force acting on it divided by its mass.

It is useful to eliminate pressure by taking the curl of the above equation, expressing it terms of vorticity

$$\omega = \text{curl } v.$$

The identity

$$\nabla\left(\frac{1}{2}v^2\right) = v \cdot \nabla v + v \times \omega, \tag{2}$$

gives,

$$\frac{\partial \omega}{\partial t} + \text{curl } \left[\omega \times v\right] = \nu \, \nabla^2 \omega\,. \tag{3}$$

By multiplying (3) by the velocity potential ψ (i.e., $v = \text{curl } \psi$) and integrating, we can get an equation for the rate of dissipation of energy. Using

$$\psi \cdot \omega = v^2 + \text{total divergence}$$

and

$$\nabla^2 \omega = -\text{curl}^2 \omega, \quad \psi \cdot \nabla^2 \omega = -\omega^2 + \text{total divergence}\,,$$

we get

$$\frac{d}{dt} \int \frac{1}{2} v^2 d^3x = -\nu \int \omega^2 d^3 x\,.$$

Here, $\int \frac{1}{2} v^2 d^3 x$ is the kinetic energy divided by density. An incompressible fluid has no potential energy.

2 Lie Algebra of Vector Fields

The phase space (the set of all initial conditions) of incompressible fluid mechanics is the space of vector fields of zero divergence. Recall that a vector field should be thought of as a first order differential operator acting on scalar fields

$$v(f) = v \cdot \nabla f\,.$$

A natural operation is the commutator of such operators,

$$[u, v] = u \cdot \nabla v - v \cdot \nabla u\,,$$

which turns the space of vector fields into a Lie algebra: it is obviously antisymmetric and can be verified to satisfy the Jacobi identity

$$[[u, v]\,, w] + [[v, w]\,, u] + [[w, u]\,, u] = 0\,.$$

Moreover, the identity

$$\mathrm{curl}(u \times w) = [w, u] + u\ \mathrm{div} w - w\ \mathrm{div} u$$

shows that the commutator of two incompressible vector fields again has zero divergence. That is, such vector fields form a Lie algebra as well. A more transparent form of the NS equations are thus,

$$\frac{\partial \omega}{\partial t} + [v, \omega] = \nu\ \nabla^2 \omega\,. \tag{4}$$

This Lie algebra of incompressible vector fields $\mathfrak{s}(3)$ is infinite dimensional, yet has remarkable analogies with the more familiar Lie algebra of rotations $\mathfrak{so}(3)$.

Underlying this is a deep analogy between the hamiltonian dynamics of a rigid body and that of an incompressible fluid. It is remarkable that the non-dissipative versions of the equations of motion of these two systems were found by Euler. He could not have known about Lie algebras; but surely, Euler must have noticed these analogies between them.

3 Ideal Incompressible Fluid vs. the Rigid Body

The equations of an ideal fluid (zero viscosity) are thus entirely determined by the commutator and the curl operator. The vorticity form of Euler equations are

$$\frac{\partial \omega}{\partial t} + [v, \omega] = 0, \quad \omega = \mathrm{curl}\, v\,.$$

3.1 The Rigid Body

For comparison recall the Euler equations for a rigid body:

$$\frac{dL}{dt} + \Omega \times L = 0, \quad L = G\Omega\,,$$

where Ω is the angular velocity and L is the angular momentum. They are related by G, the moment of inertia: a symmetric matrix that depends on the mass and the shape of the rigid body. The cross product of vectors is anti-symmetric and satisfies the Jacobi identity

$$(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = 0\,.$$

Thus it turns the three-dimensional space of vectors into a Lie algebra. This is just $\mathfrak{so}(3)$, the Lie algebra of infinitesimal rotations.

The scalar product of vectors ("dot product") satisfies the condition of being an invariant inner product of the Lie algebra $\mathfrak{so}(3)$:

$$a \cdot (b \times c) + (b \times a) \cdot c = 0\,.$$

Setting $a = \Omega, b = L, c = L$ gives us $L \cdot (\Omega \times L) = 0$. This leads to a conserved quantity, the quadratic Casimir implied by the invariant inner product

$$\frac{d}{dt}\left(\frac{1}{2} L \cdot L\right) = 0\,.$$

In addition, energy is conserved

$$\frac{dE}{dt} = 0, \quad E = \frac{1}{2}\Omega \cdot L\,.$$

Indeed, the rigid body is a hamiltonian system with the energy as the hamiltonian. To get the Poisson brackets, recall that the universal envelope of any Lie algebra is a Poisson algebra [5]. Put another way, the functions on the dual of a Lie algebra is a Poisson algebra.

Because of the invariant inner product, we do not need to distinguish between the Lie algebra and its dual. So the Poisson bracket can be written in terms of the cross and dot products as

$$\{f, g\} = L \cdot (\nabla_L f \times \nabla_L g)\ .$$

The time derivative of any observable (function of L) is its Poisson bracket with energy:

$$\{E, f\} = L \cdot (\Omega \times \nabla_L f) = -(\Omega \times L) \cdot \nabla_L f = \frac{df}{dt}\ .$$

3.2 The Ideal Incompressible Fluid

The analogue of angular velocity is the velocity field v of the fluid. It must satisfy the condition of incompressibility $\mathrm{div}\ v = 0$. The analogue of angular momentum is vorticity. They are related by the linear differential equation $\omega = \mathrm{curl}\ v$. Thus the partial differential operator curl is the analogue of moment of inertia. It can be thought of as an infinite dimensional symmetric matrix on the vector space of incompressible vector fields. Unlike the moment of inertia, it is not positive: parity will reverse its sign. It is non-degenerate: the only solution to $\mathrm{curl}\, u = 0$ within the space of incompressible vector fields is $u = 0$. (We are assuming that all our vector fields vanish at infinity faster than any power law; this allows integrations by part etc.)

The space of incompressible vector fields in three dimensions admits an invariant inner product. (This is not necessarily true in higher dimensions. Luckily, physically interesting fluids flow in three dimensional space.) This scalar product is not as obvious as for $\mathfrak{so}(3)$. But, using the identity

$$[u, w] = \mathrm{curl}\,(w \times u)$$

for incompressible vector fields, we can check that the scalar product [1]

$$\langle u, w\rangle = \int u \cdot \mathrm{curl}^{-1} w dx$$

is indeed invariant:

$$\langle u, [w, s]\rangle = \int u \cdot (s \times w)\, dx = -\langle [w, u]\,, s\rangle\,.$$

It follows immediately that there is a quadratic Casimir that is conserved in ideal incompressible fluid flow:

$$\mathcal{H} = \frac{1}{2}\int v \cdot \omega dx = \frac{1}{2}\langle \omega, \omega\rangle\,.$$

This quantity is called Helicity. It measures [6] the average linking number of vortex lines (i.e., integral curves of vorticity).

We summarize the analogy between the rigid body and the ideal incompressible fluid as a table.

	Rigid body	Ideal Incompressible fluid
velocity	Ω	v
(angular) momentum	L	ω
inertia	G	curl
phase space	$\mathfrak{so}(3) \equiv \mathbb{R}^3$	$\mathfrak{s}(3) = \{\mathbf{v}\lvert\ \mathrm{div}\ \mathbf{v}{=}0\}$
Lie product	$a \times b$	$[A, B] = A \cdot \nabla B - B \cdot \nabla A$
invariant inner product	$a \cdot b$	$\langle u, w\rangle = \int u \cdot \mathrm{curl}^{-1} w dx$
equation of motion	$\frac{dL}{dt} + \Omega \times L = 0$	$\frac{\partial\omega}{\partial t} + [v, \omega] = 0$
energy	$E = \frac{1}{2}\Omega \cdot L$	$E = \frac{1}{2}\int v \cdot v dx = \frac{1}{2}\langle v, \omega\rangle$
Casimir	$C = \frac{1}{2}L \cdot L$	$\mathcal{H} = \frac{1}{2}\int v \cdot \omega dx = \frac{1}{2}\langle \omega, \omega\rangle$
Poisson bracket	$\{f, g\} = L \cdot (\nabla_L f \times \nabla_L g)$	$\{f, g\} = \langle \omega, [\nabla_\omega f, \nabla_\omega g]\rangle$ $= \int \omega \cdot (\frac{\delta f}{\delta v} \times \frac{\delta g}{\delta v})dx$

4 Abelian Gauge Fields are Dual to Incompressible Flows

There is a duality between incompressible flows and three dimensional abelian gauge fields that is important to mention. We will see that under this duality, the helicity is just the Chern-Simons invariant of the gauge field.

Given a vector field v and a 1-form A we can form a scalar field by contraction $i_v A$. Given also a density (equal to one in our choice of co-ordinates) we can integrate this scalar field to get a number: $\int i_v(A)dx$. In terms of components this is just $\int v^i A_i dx$. If we were to change the 1-form by an abelian gauge transformation $A \to A + d\Lambda$

$$\int v^i A_i dx \mapsto \int v^i A_i dx + \int v^i \partial_i \Lambda dx \,.$$

Since $\partial_i v^i \equiv \text{div } v = 0$, the second term is zero by integration by parts. Thus, the dual of the space of incompressible vector fields is the space of 1-forms modulo gauge transformations. The action of the Lie algebra $\mathfrak{s}(3)$ on such gauge fields is the co-adjoint representation. The Chern-Simons invariant

$$\int A \wedge dA$$

is clearly invariant under the action of the group of co-ordinate transformations (vector fields are infinitesimal co-ordinate transformations). It is also invariant under gauge transformations. In the notation of the last section, this is just $\int A \cdot \text{curl } A \; dx$. This is the contra-variant version (inverse of the covariant version) of the inner product on the Lie algebra: the Chern-Simons integral, is an invariant inner product in the dual of the Lie algebra.

5 Dissipation and Helicity

So far we have ignored dissipation in our consideration of helicity. Just as total angular momentum can be dissipated by friction in a rigid body, conservation of helicity is destroyed by viscosity. From (3) we can derive

$$\frac{d\mathcal{H}}{dt} = -\nu \int \omega \cdot \text{curl } \omega dx$$

analogous to the equation for the dissipation of energy. An important difference is that helicity is not positive, nor is its time derivative. Unlike energy, it is not necessary that its magnitude should decrease with time due to viscosity. We can put a bound on the magnitude of helicity using the Cauchy-Schwarz inequality

$$\left| \frac{1}{2} \int v \cdot \omega dx \right| \leq \sqrt{\frac{1}{2} \int v^2 dx} \sqrt{\frac{1}{2} \int \omega^2 dx} \,.$$

To put a finer point on this, we consider the (analogue of the Bogomolny) inequality

$$\frac{1}{2} \left[v \pm \lambda \omega \right]^2 \geq 0 \,,$$

where λ is some quantity with the dimensions of length. That is,

$$\frac{1}{2}v^2 \pm \lambda v \cdot \omega + \lambda^2 \frac{1}{2}\omega^2 \geq 0\,.$$

Integrate and use the dissipation equation for energy:

$$E \pm 2\lambda\mathcal{H} - \frac{\lambda^2}{2\nu}\frac{dE}{dt} \geq 0\,.$$

Multiply by $\frac{2\nu}{\lambda^2 E}$ and rearrange to get the differential inequality:

$$\frac{d\log E}{dt} \leq \frac{2\nu}{\lambda^2} - \frac{4\nu}{\lambda}\frac{|\mathcal{H}|}{E}\,.$$

We can integrate this to get:

$$\log\frac{E(t)}{E_0} \leq \frac{2\nu}{\lambda^2}t - \frac{4\nu}{\lambda}\int_0^t \frac{|\mathcal{H}(t')|}{E(t')}dt'\,.$$

Exponentiating,

$$E(t) \leq E_0 e^{\frac{2\nu}{\lambda^2}t} \exp\left(-\frac{4\nu}{\lambda}\int_0^t \frac{|\mathcal{H}(t')|}{E(t')}dt\right)\,,$$

In the absence of viscosity, this is not a useful inequality: it says that energy grows slower than an exponential. We already know that in this case it is a constant! But in the presence of viscosity, this could be useful. Helicity could grow in magnitude even with viscosity: the flow can become more intertwined with time. So $\frac{|\mathcal{H}(t)|}{E(t)}$ can become large: energy decreases while the magnitude of helicity can increase. The inequality says that energy must decrease faster when the magnitude of helicity grows in comparison to energy. Perhaps this is useful in controlling the growth of potential singularities in the solutions of Navier-Stokes equations.

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