Fock Space Dualities

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Abstract. Several cases of Fock space duality occurring in the theory of many-body systems in general and nuclei in particular are discussed. All of them are special cases of a general duality theorem proved in mathematics by Howe in the 1970s. Dualities on a fermion Fock space between orthogonal Lie algebras and related groups, including an o–Pin duality recently discovered by the author, present a nice, symmetric pattern.

KEY WORDS: Fock space, classical groups, duality.

1 Introduction

Fock space dualities first appeared in physics in the context of the exploration of the new quantum mechanics in the late 1920s. Wigner and von Neumann showed [1] that, by total antisymmetry, the wave function

$$\phi((m_{l1}, m_{s1}), \dots, (m_{ln_{\mathrm{el}}}, m_{sn_{\mathrm{el}}})) \tag{1}$$

of n_{el} electrons in an atomic shell with azimutal quantum number l, where m_l and m_s are the orbital and spin magnetic quantum numbers, can be expanded on terms of the form

$$\sum_{\nu} \chi_{\nu}^{\lambda}(m_{l1}, \dots, m_{ln_{\mathrm{el}}}) \psi_{\nu}^{\lambda}(m_{s1}, \dots, m_{sn_{\mathrm{el}}}). \tag{2}$$

Here the functions χ_{ν}^{λ} and ψ_{ν}^{λ} form conjugate bases for conjugate irreducible representations (irreps) of the groups of permutations of their arguments. That is, the matrix representing a given permutation s in one irrep and the inverse transpose of that representing it in the other one are identical when s is even and differ by a factor -1 when s is odd. The equivalence classes of these irreps are described by conjugate Young diagrams λ and $\tilde{\lambda}$ like the following pair, where $\lambda = (2, 2, 1, 1, 1)$ in terms of row lengths.

$$\lambda = \;\boxed{} \tilde\lambda = \;\boxed{}$$

The Young diagram λ can have no more than $2l + 1$ rows and no more than 2 columns. Its area equals n_{el}. By the correspondence between symmetry and equivalence class of irrep of the general linear group $\mathrm{GL}(n)$ of non-singular linear transformations of an n-dimensional vector space over the complex numbers, or equivalently, its Lie algebra $\mathfrak{gl}(n)$, the functions (2) carry products of irreps of $\mathfrak{gl}(2l + 1)$, acting on the arguments m_l, and $\mathfrak{gl}(2)$, acting on the arguments m_s, and the equivalence classes of these irreps are described by the Young diagrams λ and $\tilde\lambda$.

The atomic shell is a Fock space. The general Fock space is associated with k kinds of particle, where each kind may be a kind of fermion or a kind of boson. The particles inhabit a common single-kind state space of dimension d. I label the particle kinds by letters $\tau, \upsilon, \ldots$ and basic single-kind states by letters $p, q, \ldots$. The creation operator of a particle of kind τ in the state $|p\rangle$ is denoted by $a_{p\tau}^\dagger$ and the corresponding annihilation operator by $a_{p\tau}$. These operators obey the usual commutation relations, but contrary to quantum mechanical conventions, $a_{p\tau}^\dagger$ and $a_{p\tau}$ are *not assumed Hermitian conjugate*. I impose, indeed, no Hermitian inner product on any state space. The *Fock space* of the system thus described is spanned by the states generated from the vacuum by the operators

$$1, \quad a_{p\tau}^\dagger, \quad a_{p\tau}^\dagger a_{q\upsilon}^\dagger, \quad \ldots \tag{3}$$

It has finite dimension in the case of only fermions and infinite dimension in the presence of bosons. In the atomic example, electrons with spins up and down are viewed as different kinds and the single-kind state space is spanned by the states $|m_l\rangle$. The following is then an alternative formulation of the observations above.

$\mathfrak{gl}(2l + 1)$–$\mathfrak{gl}(2)$ *duality: The Fock space of the atomic shell has the decomposition*

$$\Phi = \bigoplus \mathrm{X}_\lambda \otimes \Psi_\lambda, \tag{4}$$

where the sum runs over all Young diagrams λ with at most $2l + 1$ rows and at most 2 columns and X_λ and Ψ_λ carry irreps of $\mathfrak{gl}(2l + 1)$ and $\mathfrak{gl}(2)$ described by the Young diagrams λ and $\tilde\lambda$ and acting on the variables m_l and m_s, respectively.

Similar decompositions apply to many different Fock spaces, some of which I describe in this paper. A structure like (4) is called a *duality* in the literature.

2 Symplectic and Orthogonal Groups and Lie Algebras and Their Number Conserving and Number Non-Conserving Realisations on Fock Spaces

The symplectic group $\mathrm{Sp}(n)$ and orthogonal group $\mathrm{O}(n)$ are subgroups of $\mathrm{GL}(n)$ defined by non-degenerate bilinear forms b. In terms of basic vectors $|i\rangle$ for the defining vector space of $\mathrm{GL}(n)$, their members g obey

$$\sum_{kl}\langle b|kl\rangle\langle k|g|i\rangle\langle l|g|j\rangle = \langle b|ij\rangle. \tag{5}$$

When b is *skew symmetric*, the group is *symplectic*, when b is *symmetric*, it is *orthogonal*. For a given n, all symplectic groups are isomorphic and all orthogonal groups are isomorphic, independently of b. I denote the Lie algebras of $\mathrm{Sp}(n)$ and $\mathrm{O}(n)$ by $\mathfrak{sp}(n)$ and $\mathfrak{o}(n)$. The group $\mathrm{Sp}(n)$ only exists for even n, the group $\mathrm{O}(n)$ for every n. While $\mathrm{Sp}(n)$ is simply connected, $\mathrm{O}(n)$ is not even connected and its maximal connected subgroup $\mathrm{SO}(n)$ of index 2 formed by the orthogonal transformations with determinant 1 not simply connected. (The transformations in the coset have determinant -1.) Both $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ have double covering groups $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$, where the latter is simply connected and a subgroup of the former of index 2. The equivalence classes of irreps of $\mathfrak{sp}(n)$ and $\mathfrak{o}(n)$ correspond 1–1 to those of the simply connected groups $\mathrm{Sp}(n)$ and $\mathrm{Spin}(n)$. The *spin irreps* of $\mathrm{Spin}(n)$ are double valued on $\mathrm{SO}(n)$ and likewise those of $\mathrm{Pin}(n)$ on $\mathrm{O}(n)$. The equivalence classes of $\mathfrak{sp}(n)$ and $\mathfrak{o}(d)$ irreps are described by Young diagrams with at most $n/2$ rows. The $\mathfrak{sp}(n)$ Young diagrams are ordinary ones [2]. To describe spin $\mathfrak{o}(n)$ irreps (which expand to spin irreps of $\mathrm{Spin}(n)$) one also needs Young diagrams with half-integral row lengths like the following with row lengths $9/2, 7/2, 3/2$.
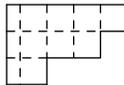
Further, for even n, the Young diagrams of maximal depth come in pairs with opposite signs of the "lengths" of their bottom rows like the following pair for $n = 6$ with row lengths $4, 3, \pm 2$.
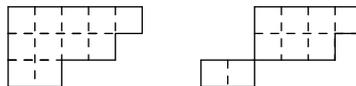
I call *mirrors* the associated equivalence classes of $\mathfrak{o}(d)$ irreps. For $n = 2$ the edge whence the single row extends must be specified. (The abelian Lie algebra $\mathfrak{o}(2)$ actually has a continuum of inequivalent 1-dimensional irreps. Those described by the present Young diagrams are the only ones that occur in the theorems below.) [3]

The single-kind state space may be taken as defining vector space for $\mathfrak{sp}(d)$ and

$\mathfrak{o}(d)$. These Lie algebras then have realisations on any Fock space given by

$$g \mapsto \sum_{pq\tau} \langle p|g|q \rangle a_{p\tau}^\dagger a_{p\tau} \tag{6}$$

for an arbitrary member g of the Lie algebra. These operators are seen to conserve the number of particles. I reserve the symbols $\mathfrak{sp}(d)$ and $\mathfrak{o}(d)$ for these realisations, which I call *number conserving*. For systems of only fermions or only bosons, the Lie algebras $\mathfrak{sp}(2k)$ and $\mathfrak{o}(2k)$ have realisations spanned by the operators

$$\sum_p a_{p\tau}^\dagger a_{p\upsilon} \mp \frac{d}{2}, \qquad \sum_{pq} \langle pq|b \rangle a_{p\tau}^\dagger a_{q\upsilon}^\dagger, \qquad \sum_{pq} \langle b|pq \rangle a_{p\tau} a_{q\upsilon} \tag{7}$$

with $-$ for fermions and $+$ for bosons. Here the matrix element $\langle pq|b \rangle$ of the dual bilinear form is defined by

$$\sum_r \langle b|pr \rangle \langle qr|b \rangle = \delta_{pq}. \tag{8}$$

The operators in the last two sets in (7) evidently do not conserve particle number. (The sets are empty for $k = 1$ in the cases of fermions and $\mathfrak{o}(2k)$ and bosons and $\mathfrak{sp}(2k)$.) I reserve the symbols $\mathfrak{sp}(2k)$ and $\mathfrak{o}(2k)$ for these realisations, which I call *number non-conserving*. The sets of operators (6) and (7) commute.

3 Helmers's Theorem and Its Orthogonal Analogon

In 1961, in the wake of the nuclear BCS theory, Helmers proved the following [4].

$\mathfrak{sp}(d)$–$\mathfrak{sp}(2k)$ *duality (Helmers): For even d, a fermion Fock space has the decomposition*

$$\Phi = \bigoplus X_\lambda \otimes \Psi_\mu, \tag{9}$$

where $\mathfrak{sp}(d)$ acts on X_λ producing an irrep with the Young diagram λ and $\mathfrak{sp}(2k)$ acts on Ψ_μ producing an irrep with the Young diagram μ. The sum runs over all such pairs (λ, μ) that λ and a rotated and reflected copy of μ fill a $k \times d/2$ rectangle without overlap as in the following example for $d = 12$ and $k = 4$, where $\lambda = (4, 3, 2, 2, 1, 1)$ and $\mu = (5, 4, 2)$.
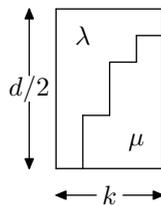
Much later, in 2019, i obtained the following orthogonal analogon of Helmers's theorem [5].

$\mathfrak{o}(d)$–$\mathfrak{o}(2k)$ *duality: A fermion Fock space has the decomposition*

$$\Phi = \bigoplus X_\lambda \otimes \Psi_\mu, \tag{10}$$

where $\mathfrak{o}(d)$ acts on X_λ producing a representation associated with the Young diagram λ and $\mathfrak{o}(2k)$ acts on Ψ_μ producing a representation associated with the Young diagram μ. The sum runs over all such pairs (λ, μ) that λ and a rotated and reflected copy of μ fill a $k \times d/2$ rectangle without overlap as in the following example for $d = 11$ and $k = 4$, where $\lambda = (4, 3, 2, 2, 1)$ and $\mu = (9/2, 7/2, 3/2, 1/2)$.
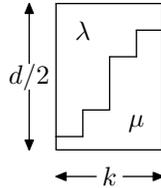
If the border between λ and the copy of μ hits the left edge of the rectangle (as in the example), the $\mathfrak{o}(d)$ representation is irreducible while the $\mathfrak{o}(2k)$ representation is the direct sum of mirror irreps. If the border hits the bottom edge (which requires that d is even), the $\mathfrak{o}(2k)$ representation is irreducible while the $\mathfrak{o}(d)$ representation is the direct sum of mirror irreps.

The proofs of both theorems are based on comparison of the characters on both sides of the equations (9) and (10).

Several special cases of these theorems have applications in nuclear physics. Thus for $k = 1$, the number non-conserving Lie algebra $\mathfrak{sp}(2k)$ is closely related to Kerman's *quasispin* algebra [6], usually described as an $\mathfrak{su}(2)$ algebra. The latter is a Lie algebra over the reals whose complexification is isomorphic to $\mathfrak{sp}(2)$. The $\mathfrak{sp}(d)$–$\mathfrak{sp}(2)$ duality connects the quantum numbers of quasispin and *seniority* as exploited extensive in the work of Talmi, in particular [7]. For $k = 2$, the Lie algebra $\mathfrak{sp}(2k) = \mathfrak{sp}(4)$ is identical to the Lie algebra $\mathfrak{o}(5)$ introduced by Flowers and Szpikowski to describe systems with both neutrons and protons [8]. For $k = 4$, corresponding to the 4-dimensional space of the nucleonic spin and isospin, $\mathfrak{o}(2k)$ is the Lie algebra $\mathfrak{o}(8)$ advanced by these authors as a "quasispin algebra for LS coupling" [9]. Both Lie algebras $\mathfrak{o}(5)$ and $\mathfrak{o}(8)$ attracted attention around the turn of millennium in discussions of pairing in nuclei including isospin $T = 0$ paring.

4 Howe's Theorem

All dualities mentioned so far are special cases of a very general theorem proved in mathematics by Howe in the 1970s [10]. Presenting his theorem requires more definitions. The following formalism is not that of Howe but a "physicists version" based on second quantisation [3]. The Fock space is the general one with possibly both fermion and bosons. Let G be a "classical" group [2], $\mathrm{GL}(d)$, $\mathrm{Sp}(d)$ or $\mathrm{O}(d)$, with the single-kind state space as the defining vector space. It has a realisation on the Fock space, which I denote by the same symbol as the abstact group. I denote by g an arbitrary member of G and by γ its realisation on the Fock space. For a given τ, the realisation may be either *cogredient*, that is,

$$\gamma a_{p\tau}^\dagger = \left(\sum_q \langle q|g|p\rangle a_{q\tau}^\dagger \right) \gamma, \quad \gamma a_{p\tau} = \left(\sum_q \langle p|g^{-1}|q\rangle a_{q\tau} \right) \gamma. \tag{11}$$

for every g, or *contragredient*, that is,

$$\gamma a_{p\tau}^\dagger = \left(\sum_q \langle p|g^{-1}|q\rangle a_{q\tau}^\dagger \right) \gamma, \quad \gamma a_{p\tau} = \left(\sum_q \langle q|g|p\rangle a_{q\tau} \right) \gamma. \tag{12}$$

for every g. In the cases of $\mathrm{Sp}(d)$ and $\mathrm{O}(d)$, co- and contragredient actions are equivalent because the bilinear form b provides a similarity transformations between the matrices $\langle p|g|q\rangle$ and $\langle q|g^{-1}|p\rangle$.

Acting on the operators $a_{p\tau}^\dagger$ and $a_{p\tau}$, I define a bracket $|\cdot,\cdot|$ which equals the anticommutator $\{\cdot,\cdot\}$ when both operands are fermion operators, otherwise the commutator $[\cdot,\cdot]$, so that their commutation relations can be written

$$|a_{p\tau}^\dagger, a_{q\upsilon}^\dagger| = |a_{p\tau}, a_{q\upsilon}| = 0, \quad |a_{p\tau}, a_{q\upsilon}^\dagger| = \delta_{p\tau,q\upsilon}. \tag{13}$$

I define another bracket $]\cdot,\cdot[$ which is the complete opposite, being equal to the commutator when both operands are fermion operators and otherwise the anti-commutator. Both brackets can be extended to the span $\mathfrak{A}$ of the set of creation and annihilation operators. The bracket $|\cdot,\cdot|$ can be extended further to the set

$$\bar{\mathfrak{h}} = \operatorname{span}\{\, ab \mid a, b \in \mathfrak{A} \,\}, \tag{14}$$

which, by the commutation relations, includes the numbers. By definition, $|ab, cd| = [ab, cd]$ when either both a and b or both c and d are fermion operators or both of them are boson operators, and $|ab, cd| = \{ab, cd\}$ when both ab and cd are products of one boson operator and one fermion operator. One can check that this defines $|\cdot,\cdot|$ unambiguously as a bilinear product on $\bar{\mathfrak{h}}$ and that $\bar{\mathfrak{h}}$ is closed under the action of $|\cdot,\cdot|$. In particular $|h_1, h_2| = [h_1, h_2] = 0$ when any one of h_1 and h_2 is a number. The set $\bar{\mathfrak{h}}$ equipped with the bilinear product

$|\cdot, \cdot|$ forms a so-called *Lie superalgebra*. When only fermions or only bosons are present, it becomes an ordinary Lie algebra. The set

$$\mathfrak{h} = \mathrm{span}\{\,]a, b[\ |\ a, b \in \mathfrak{A}\,\} \tag{15}$$

can be shown to be a subalgebra of $\bar{\mathfrak{h}}$. Also the pointwise G invariant subset of $\mathfrak{h}$, that is,

$$\mathfrak{h}^G = \{\, h \in \mathfrak{h} \mid \gamma h = h \gamma \quad \forall g \in G \,\} \tag{16}$$

is a subalgebra because, as a subgroup of $\mathrm{GL}(d)$, the group G preserves the bracket $|\cdot, \cdot|$ on $\mathfrak{A}$. (The members of $\mathrm{GL}(d)$ just change the basis for the single-kind state space.) Explicitly,

1. $\mathfrak{h}^{\mathrm{GL}(d)}$ is spanned by the operators

$$\sum_p]a_{p\tau}^\dagger, a_{p\upsilon}[, \quad (\tau, \upsilon) \in K \times K \cup \bar{K} \times \bar{K},$$

$$\sum_p]a_{p\tau}^\dagger, a_{p\upsilon}^\dagger[, \quad \sum_p]a_{p\tau}, a_{p\upsilon}[, \quad (\tau, \upsilon) \in K \times \bar{K}, \tag{17}$$

where K and $\bar{K}$ are the sets of τ with co- and contragredient actions of $\mathrm{GL}(d)$.

2. $\mathfrak{h}^{\mathrm{Sp}(d)}$ and $\mathfrak{h}^{\mathrm{O}(d)}$ are spanned by the operators

$$\sum_p]a_{p\tau}^\dagger, a_{p\upsilon}[, \quad \sum_{pq} \langle pq|b \rangle\,]a_{p\tau}^\dagger, a_{q\upsilon}^\dagger[, \quad \sum_{pq} \langle b|pq \rangle\,]a_{q\tau}, a_{p\upsilon}[. \tag{18}$$

Howe's theorem now reads:

General duality (Howe): The general Fock space has a decomposition

$$\Phi = \bigoplus_\lambda \mathrm{X}_\lambda \otimes \Psi_\lambda,$$

where G acts irreducibly on X_λ and $\mathfrak{h}^G$ acts irreducibly on Ψ_λ. For $\lambda \neq \mu$, the representations of G on X_λ and X_μ are inequivalent and the representations of $\mathfrak{h}^G$ on Ψ_λ and Ψ_μ are inequivalent. The spaces X_λ have finite dimensions.

Howe's proof is based in the so-called *first main theorem* [2] of the classical groups, which states that the algebra of their invariants is generated by the quadratic invariants. Special cases include fermion and boson $\mathrm{GL}(d)$–$\mathfrak{gl}(k)$ dualities, fermion $\mathrm{Sp}(d)$–$\mathfrak{sp}(2k)$ and $\mathrm{O}(d)$–$\mathfrak{o}(2k)$ dualities and boson $\mathrm{Sp}(d)$–$\mathfrak{o}(2k)$ and $\mathrm{O}(d)$–$\mathfrak{sp}(2k)$ dualities. The boson $\mathfrak{o}(2k)$ and $\mathfrak{sp}(2k)$ irreps have infinite dimensions. In particular, the Lie algebra of the $\mathrm{O}(d)$–$\mathfrak{sp}(2k)$ duality is known in nuclear physics in the case $d = A$ and $k = 3$, where A is the mass number, as the complexification of the Lie algebra "$\mathrm{Sp}(3, \mathbb{R})$" suggested by Rosensteel and Rowe [11] to model nuclear collective motion. Since we are in a boson environment, its irreps are infinite-dimensional. By Howe's theorem, they can be labelled by the known equivalence classes of finite-dimensional irreps of $\mathrm{O}(A)$.

5 Relation to the Dualities with Pairs of Lie Algebras

Unlike the dualities in Sections 1 and 3, which relate a couple of Lie algebras, Howe duality relates a group and a Lie (super-)algebra. Howe's theorem does not specify the relation of the equivalence classes of the irreps carried by X_λ and Ψ_λ. This matter is addressed in later papers by Howe [12], Rowe, Repka and Carvalho [13] and me [3]. It is fairly easy to show that in the special cases with $GL(d)$ and $Sp(d)$, the distinction between the group and its Lie algebra does not matter. (The case of $GL(d)$ is the most complicated one; see [3].) The case of $O(d)$ is more involved due to the more complicated topology of this group. The equivalence classes of $O(n)$ irreps were identified by Weyl [2]. They are described by ordinary Young diagrams subject to the constraint that no pair of different columns have depths whose sum exceeds n. Rowe, Repka and Carvalho obtained the relation between the equivalence classes of the $O(d)$ and $\mathfrak{o}(2k)$ irreps in the Howe $O(d)$–$\mathfrak{o}(2k)$ duality from an analysis of highest weight states [13]. In 2020, I derived it from the $\mathfrak{o}(d)$–$\mathfrak{o}(2k)$ duality theorem in Section 3 and described it diagrammatically as follows [3].

$O(d)$–$\mathfrak{o}(2k)$ *duality: A fermion Fock space has the decomposition*

$$\Phi = \bigoplus X_\lambda \otimes \Psi_\mu, \tag{19}$$

where $O(d)$ acts on X_λ producing an irrep with the Young diagram λ and $\mathfrak{o}(2k)$ acts on Ψ_μ producing an irrep with the Young diagram μ. The sum runs over the all such pairs (λ, μ) that λ and a rotated and reflected copy of μ fill a $k \times d/2$ rectangle without overlap provided a part of a row in μ of negative length cancels a part of λ extruding the rectangle as in the following example for $d = 11$ and $k = 4$, where $\lambda = (4, 3, 2, 2, 1, 1, 1)$ and $\mu = (9/2, 7/2, 3/2, -3/2)$.
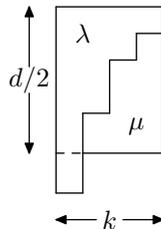
The symmetric way in which $\mathfrak{o}(d)$ and $\mathfrak{o}(2k)$ enter the $\mathfrak{o}(d)$–$\mathfrak{o}(2k)$ duality theorem suggests that one might similarly derive from this theorem a duality between $\mathfrak{o}(d)$ and a group. I recently showed that this is indeed true and that the group is $Pin(2k)$ [14]. I first constructed a realisation of $Pin(2k)$ on the fermion Fock space from the observation that $Pin(2k)$ is realised within the Clifford algebra $Cl(k)$ [15], which is realised, in turn, by the algebra generated by the operators $a_{1\tau}^\dagger$ and $a_{1\tau}$. More precisely, $Pin(2k)$ can be identified with the set of products of linear combinations $s = \sum_\tau (\alpha_\tau a_{1\tau}^\dagger + \beta_\tau a_{1\tau})$ obeying $s^2 = -1$ and $Spin(2k)$ with the set of products of an even number of factors of this form [16]. A single

s thus connects the subgroup $\mathrm{Spin}(2k)$ and its coset. In particular, in my realisation on the fermion Fock space, $s = a_{11}^\dagger - a_{11}$ maps to a "partial particle-hole conjugation" σ obeying

$$\sigma a_{p1} = (-)^d a_{p1}^\dagger \sigma, \quad \sigma a_{p1}^\dagger = (-)^d a_{p1}\sigma,$$
$$\sigma a_{p\tau} = (-)^d a_{p\tau}\sigma, \quad \sigma a_{p\tau}^\dagger = (-)^d a_{p\tau}^\dagger \sigma, \quad \tau > 1. \quad (20)$$

One can *define* a set of generalised Young diagrams by the following rules. (i) The rows have either integral or half-integral, positive row lengths, which decrease weakly from top to bottom. (ii) If the row lengths are integral, no pair of different columns have depths whose sum exceeds $2k$. (iii) If the row lengths are half-integral, the Young diagram has exactly k rows. One can assign to each such Young diagram properties of a $\mathrm{Pin}(2k)$ irrep, which my limited space does not allow me to specify. (See ref. [14] for details.) Essentially, the Young diagrams with integral row lengths describe $\mathrm{Pin}(2k)$ irreps which factor through $\mathrm{O}(2k)$, yielding an $\mathrm{O}(2k)$ irrep with the same diagram, while those with half-integral row lengths describe $\mathrm{Pin}(2k)$ irreps which split upon restriction into a pair of mirror $\mathfrak{o}(2k)$ irreps, one of which has the same Young diagram. My constructed realisation of $\mathrm{Pin}(2k)$ on the fermion Fock space being denoted by the same symbol, I then proved the following.

$\mathfrak{o}(d)$–$\mathrm{Pin}(2k)$ *duality: A fermion Fock space has the decomposition*

$$\Phi = \bigoplus X_\lambda \otimes \Psi_\mu,$$

where $\mathfrak{o}(d)$ acts on X_λ producing an irrep with the Young diagram λ and $\mathrm{Pin}(2k)$ acts on Ψ_μ producing an irrep with the Young diagram μ. The sum runs over all such pairs (λ, μ) that λ and a rotated and reflected copy of μ fill a $k \times d/2$ rectangle without overlap provided a part of a row in λ of negative length cancels a part of μ extruding the rectangle as in the following example for $d = 12$ and $k = 4$, where $\lambda = (4, 3, 2, 2, 1, -1)$ and $\mu = (5, 4, 2, 1, 1)$.
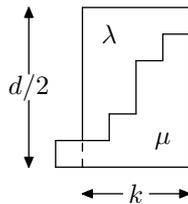
6 Concluding Remarks

The triple of the $\mathfrak{o}(d)$–$\mathfrak{o}(2k)$, $\mathrm{O}(d)$–$\mathfrak{o}(2k)$ and $\mathfrak{o}(d)$–$\mathrm{Pin}(2k)$ dualities seem to present a nice, unified picture with a high degree of symmetry between the number conserving and the number non-conserving groups or Lie algebras. The

only asymmetry is O(d) versus Pin($2k$), or equivalently, the presence of only non-spin irreps of $\mathfrak{o}(d)$. This asymmetry is related to $2k$ being always even while both even and odd values are allowed for d. My proofs of the O(d)–$\mathfrak{o}(2k)$ and $\mathfrak{o}(d)$–Pin($2k$) dualities is based on the $\mathfrak{o}(d)$–$\mathfrak{o}(2k)$ duality, whose proof is based, in turn, on a comparison of characters. In particular the proof of the $\mathfrak{o}(d)$–Pin($2k$) duality does not require a Pin(n) first main theorem, and in fact, such a theorem is not known [17]. Application of the method is restricted, though, to the fermion case because it employs Weyl's formula [18] for the character of a finite-dimensional irrep of a semi-simple Lie algebra over the complex numbers.

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