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Multiple $SO(5)$ Isovector Pairing and Seniority $Sp(2\Omega)$ Multi-j Algebras with Isospin

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Abstract. With nucleons occupying several shell model j orbits, the isovector pair creation operator A^1_μ (creates a two particle state with angular momentum $J = 0$ and isospin $T = 1$) is no longer unique. Choosing it to be a sum of single-j isovector pair creation operators each with a phase, there will be multiple pair $SO(5)$ algebras with isospin; with r number of j orbits, there will be 2^{r-1} $SO(5)$ algebras each with a corresponding complementary $Sp(2\Omega)$ algebra [$2\Omega = \sum_j (2j+1)$] that gives seniority and reduced isospin quantum numbers. Three applications of multiple $SO(5)$ algebras are presented demonstrating the usefulness of considering $SO(5)$ pairing algebras with general sign factors.

KEY WORDS: pairing, multiple algebras, multi-j, $SO(5)$, $Sp(2\Omega)$.

1 Introduction

With identical nucleons (protons or neutrons) in a single-j shell, the pair creation and annihilation operators (S_+ and S_- respectively) and the number operator ($\hat{n}$) generate remarkably the quasi-spin $SU(2)$ algebra. On the other hand, the $Sp(2j+1)$ subalgebra of the Spectrum Generating Algebra (SGA) $U(2j+1)$ is 'complementary' to the quasi-spin-$SU(2)$ algebra and the seniority quantum number v that labels the states w.r.t. $Sp(2j+1)$ algebra, i.e. irreducible representation (irreps), corresponds to the quasi-spin quantum number and similarly, particle number m that labels the irreps of $U(2j+1)$ corresponds to the z-component of quasi-spin. More importantly, this solves the pairing Hamiltonian $H_p = -S_+S_-$ and allows one to extract m dependence of many particle matrix elements of a given operator; see [1] for full details and applications.

All the single-j shell results extend to the multi-j shell situation i,e. for identical nucleons occupying several-j orbits, with $(2j+1)$ replaced by $2\Omega = \sum_j (2j+1)$. In this situation, v is called generalized seniority. Now, a new result is that with

r number of j-orbits there will be 2^{r-1} number of quasi-spin $SU(2)$ and the corresponding $Sp(2\Omega)$ algebras ($S_+ = \sum_j \alpha_j S_+(j)$; $\alpha_j = \pm 1$). These multiple quasi-spin algebras (one for each α_j choice) play an important role in deciding selection rules for electric and magnetic multipole operators. See [2] for details regarding these multiple pairing algebras and [1, 3–5] for the goodness of multi-j seniority in certain nuclei. Also, multi-j seniority provides a framework for shell model theory of the interacting boson model [6, 7].

With isospin (T) degree of freedom, the algebra changes to the more complex $SO(5)$ algebra. With m-nucleons in a single-j orbit the SGA is $U(2(2j+1))$ with the 2 coming from isospin. An isospin conserving subalgebra chain then is $U(2(2j+1)) \supset [U(2j+1) \supset Sp(2j+1)] \otimes SU_T(2)$. Particle number m labels $U(2(2j+1))$, (m,T) label $U(2j+1)$ and T labels $SU_T(2)$ irreps. It is recognized in very early years of shell model that the isovector pair creation and annihilation operators, isospin and the number operator generate a $SO(5)$ algebra, which is 'complementary' to the above $Sp(2j+1)$ algebra. The irreps of $SO(5)$ contain two labels and they can be written in terms of the seniority v and reduced isotopic spin t quantum numbers with (v,t) uniquely labeling the $Sp(2j+1)$ irreps. Another important result is that the isovector pairing Hamiltonian is simply related to the quadratic Casimir invariants of $SO(5)$ and $Sp(2j+1)$. However, an unsatisfactory aspect of the $SO(5)$ algebra of the shell model is that it does not contain isoscalar pair operators in its algebra. For the first papers on single-j shell pairing $Sp(2j+1)$ algebra and the corresponding $SO(5)$ algebra see [8–13]. Similarly, for technical work on these algebras (for example deriving analytical formulas for the Wigner coefficients of $SO(5)$) see [12, 14–18] and for recent applications see [19–23] and references therein. Although many of the single-j shell results extend to the multi-j shell systems, for the multi-j shell situation a crucial aspect is that there will be multiple $SO(5)$ algebras as the isovector pair creation operator here is no longer unique. The purpose of this paper is to introduce and analyze these multiple $SO(5)$ isovector pairing algebras with isospin.

Before proceeding further, let us add that the general mathematical theory describing complementarity between identical nucleon number non-conserving quasi-spin $SU(2)$ and the number conserving $Sp(2\Omega)$, complementarity between $SO(5)$ and $Sp(2\Omega)$ with isospin and similar complementarity between many other algebras (for example proton-neutron pairing $SO(8)$ algebra with LST coupling and the corresponding number conserving algebras [24, 25]) including those for boson systems (see for example [26, 27]) is due to Neergard based on Howe's general duality theorem [28, 29]; first proof of complementarity is due to Helmers [10] and later work is due to Rowe et al. [30]. Now we will give a preview.

In Section 2, multiple pairing $SO(5)$ and $Sp(2\Omega)$ algebras for the multi-j situation are introduced. Section 3 gives formulas for constructing many-particle matrix elements of the pairing Hamiltonian generating multiple $SO(5)$ algebras.

In Section 4 presented are some applications. Finally, Section 5 gives conclusions.

2 Multiple Multi-j Shell $SO(5)$ and $Sp(2\Omega)$ Algebras with Isospin

Consider the angular momentum zero coupled isovector pair creation operator $A^1_\mu(j)$ for nucleons in a single-j shell, $A^1_\mu(j) = \sqrt{(2j+1)/2}\,(a^\dagger_{j\frac{1}{2}} a^\dagger_{j\frac{1}{2}} r)^{0,1}_{0,\mu}$. Now, with nucleons in $(j_1, j_2, \ldots, j_r)$ orbits, pair creation operator can be taken as a linear combination of the single-j shell pair creation operators but with different phases giving the generalized isovector pairing operator to be,

$$\mathcal{A}^1_\mu(\beta) = \sum_{p=1}^{r} \beta_{j_p} A^1_\mu(j_p)\,; \quad \{\beta\} = \{\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_r}\} = \{\pm 1, \pm 1, \ldots\}\,. \tag{1}$$

Strikingly, the ten operators $\mathcal{A}^1_\mu(\beta)$, $[\mathcal{A}^1_\mu(\beta)]^\dagger$, T^1_μ and $Q_0 = [\hat{n} - 2\Omega]/2$ (equivalently $\hat{n}$) form a pairing $SO^{(\beta)}(5)$ algebra for each $\{\beta\}$ set when $\beta_{j_p} = \pm 1$ as in Eq. (1). Note that $2\Omega = \sum_j (2j+1)$, isospin generators $T^1_\mu = \sum_j \sqrt{(2j+1)/2}\,(a^\dagger_{j\frac{1}{2}} \tilde{a}_{j\frac{1}{2}})^{0,1}_\mu$ and the number operator $\hat{n} = \sum_j \sqrt{2(2j+1)}\,(a^\dagger_{j\frac{1}{2}} \tilde{a}_{j\frac{1}{2}})^{0,0}$. Without loss of generality we can choose $\beta_{j_1} = +1$ and then the remaining β_{j_p} will be ± 1. Thus, there will be 2^{r-1} $SO(5)$ algebras. Then, with two j orbits we have two $SO(5)$ algebras $SO^{(+,+)}(5)$ and $SO^{(+,-)}(5)$, with three we have four $SO(5)$ algebras $SO^{(+,+,+)}(5)$, $SO^{(+,+,-)}(5)$, $SO^{(+,-,+)}(5)$ and $SO^{(+,-,-)}(5)$, with four j orbits there will be eight $SO(5)$ algebras and so on. Significantly, the isovector pairing Hamiltonians $H_p(\beta) = -G \sum_\mu \mathcal{A}^1_\mu(\beta) \left[\mathcal{A}^1_\mu(\beta)\right]^\dagger$ with G the pairing strength, is simply related to $\mathcal{C}_2(SO^{(\beta)}(5))$, the quadratic Casimir invariant of $SO^{(\beta)}(5)$; $\mathcal{C}_2(SO^{(\beta)}(5)) = 2 \sum_\mu \mathcal{A}^1_\mu(\beta) \left[\mathcal{A}^1_\mu(\beta)\right]^\dagger + T^2 + Q_0(Q_0 - 3)$. Further, $SO(5)$ irreps are labeled by (ω_1, ω_2) with ω_1 and ω_2 both integers or half integers and $\omega_1 \geq \omega_2 \geq 0$ [31]. Then, the eigenvalues of $\mathcal{C}_2(SO^{(\beta)}(5))$ are $\left\langle \mathcal{C}_2(SO^{(\beta)}(5)) \right\rangle^{(\omega_1,\omega_2)} = \omega_1(\omega_1 + 3) + \omega_2(\omega_2 + 1)$. Expressions for Casimir invariants are given by Racah very early [32]. We will now turn to the complementary $Sp^{(\beta)}(2\Omega)$ algebras.

Consider one-body operators $u^{k,t}_{m_k,m_t}(j_1, j_2)$ defined in terms of the single particle creation and annihilation operators in (jt) space, $u^{k,t}_{m_k,m_t}(j_1, j_2) = (a^\dagger_{j_1\frac{1}{2}} \tilde{a}_{j_2\frac{1}{2}})^{k,t}_{m_k,m_t}$ where $\tilde{a}_{j-m,\frac{1}{2}-m_t} = (-1)^{j-m+\frac{1}{2}-m_t}\, a_{jm,\frac{1}{2}m_t}$. Now, it is easy to prove that the operators $u^{k,t}(j_1, j_2)$ generate the $U(4\Omega)$ SGA. Moreover, we have the subalgebra $U(4\Omega) \supset [U(2\Omega) \supset Sp(2\Omega)] \otimes SU_T(2)$ with $u^{k,0}_{m_k,0}(j_1, j_2)$ operators generating $U(2\Omega)$ and $SU_T(2)$ generating isospin. Following the results in [2, 11, 15], it is easy to recognize the generators of $Sp(2\Omega)$

and they are,

$$\begin{aligned} &u^{k,0}_{\mu,0}(j,j)\,; \quad k = \text{odd} \\ &V^{k,0}_{\mu,0}(j_1,j_2) = u^{k,0}_{\mu,0}(j_1,j_2) + X(j_1,j_2,k)\; u^{k,0}_{\mu,0}(j_2,j_1)\,; \quad j_1 > j_2\,. \end{aligned} \tag{2}$$

Now, the most important result is that for every $SO^{(\beta)}(5)$, there will be a complementary $Sp^{(\beta)}(2\Omega)$ algebra with generators given by Eq. (2) provided

$$X(j_1,j_2,k) = (-1)^{j_1+j_2+k}\,\beta_{j_1}\,\beta_{j_2}\;. \tag{3}$$

Proof for the complementarity is given first by Helmers [10]. Thus, the multiple $SO^{(\beta)}(5)$ algebras that have number non-conserving generators and $Sp^{(\beta)}(2\Omega)$ algebras with only number conserving generators are complementary provided Eq. (3) is satisfied along with Eqs. (1) and (2).

Turning to the irreps, all m nucleon states transform as the antisymmetric irrep $\{1^m\}$ of $U(4\Omega)$ and the irreps of $U(2\Omega)$ will be two columned irreps $\{2^{m_1}1^{m_2}\}$ in Young tableaux notation with $2m_1 + m_2 = m$ and $T = m_2/2$. Similarly, the $Sp(2\Omega)$ irreps are two columned denoted by $\langle 2^{v_1}1^{v_2}\rangle$ giving $v = 2v_1 + v_2$ the seniority quantum number and $t = v_2/2$ the reduced isospin. Group theory allows us to obtain $(m,T) \rightarrow (v,t)$ reductions or $(v,t) \rightarrow T$ for a given m [31]. More importantly, it can be shown that (ω_1,ω_2) are equivalent to (v,t) giving $\omega_1 = \Omega - (v/2)$ and $\omega_2 = t$. With all these, the eigenvalues of $H_p^{(\beta)}$ are [11],

$$\langle H_p^{(\beta)}\rangle^{m,T;v,t} = -\frac{G}{4}\Big[(m-v)(2\Omega+3-\frac{m+v}{2})-2T(T+1)+2t(t+1)\Big]. \tag{4}$$

Note that $SO^{(\beta)}(5) \supset [SO(3) \supset SO(2)] \otimes U(1)$ with $SO(3)$ generating T, $SO(2)$ generating M_T (T_z quantum number (N-Z)/2) and $U(1)$ generating particle number or $H_1 = (m-2\Omega)/2$. Then, the eigenstates of $H_p^{(\beta)}$ are

$$\Big|\Psi_{H_p^{(\beta)}}\Big\rangle \Rightarrow \Big|(\Omega - \frac{v}{2},t), H_1 = \frac{m-2\Omega}{2}, T, M_T = \frac{\text{N-Z}}{2}\Big\rangle \tag{5}$$

and the labels do not depend on (β). However, explicit structure of the wavefunctions do depend on (β); see Section 4. Thus, they will effect various selection rules and matrix elements of certain transition operators (see Section 4). Finally, with $SO(5)$ algebra it is possible to factorize (m,T) dependence of various matrix elements [1,15,21]. In order to enumerate the irrep labels in Eq. (5), used is $U(4\Omega) \supset [U(2\Omega) \supset Sp(2\Omega)] \otimes SU_T(2)$ reductions starting with the $\{1^m\}$ irrep of $U(4\Omega)$. All the rules for these are known [24,31,33].

3 Construction of Many-Particle Matrix for Pairing Hamiltonian Generating Multiple $SO(5)$ Algebras

In order to probe the role of multiple pair $SO^{(\beta)}(5)$ algebras with isospin, we need to obtain the eigenstates of the pairing Hamiltonian $\mathcal{H}_p$ as a function of $\{\beta\}$'s. A convenient basis for constructing the $\mathcal{H}_p$ matrix is the product basis defined by the single-j shell $SO(5)$ basis. We will illustrate this using two j-orbits say j_1 and j_2. Hereafter, we call the corresponding spaces a and b respectively (or 1 and 2). Then, the basis states are, $\Psi_{ab}(T \;\; M_T) = \left|(\omega_1^a\omega_2^a)H^aT^a, (\omega_1^b\omega_2^b)H^bT^b; T\, M_T\right\rangle$ or equivalently $|(v_1,t_1)m_1T_1,(v_2,t_2)m_2T_2;T\,M_T\rangle$. Given m number of nucleons, with m_1 in number in the first orbit and m_2 in the second orbit, $m = m_1 + m_2$. Note that $\Omega_1 = j_1 + \frac{1}{2}$, $\Omega_2 = j_2 + \frac{1}{2}$, $H^a = \frac{m_1}{2} - \Omega_1$ and $H^b = \frac{m_2}{2} - \Omega_2$. Similarly T^a and T^b are the isospins in the two spaces respectively. Now, a general pairing Hamiltonian [with $\mathcal{A}^1_\mu(\alpha) = A^1_\mu(j_1) + \alpha\, A^1_\mu(j_2)$] is,

$$\mathcal{H}_p(\xi,\alpha) = \frac{(1-\xi)}{m}\hat{n}_2 - \frac{\xi}{m^2}\Big\{4\sum_\mu \mathcal{A}^1_\mu(\alpha)\left[\mathcal{A}^1_\mu(\alpha)\right]^\dagger\Big\}. \tag{6}$$

Here, $\hat{n}_2$ is the number operator for the second orbit and ξ and α are parameters changing from 0 to 1 and $+1$ to -1 respectively. Note that for $\xi = 1$ and $\alpha = +1$ we have a $SO^{(+)}(5)$ algebra in the total two-orbit space and similarly for $\xi = 1$ and $\alpha = -1$ the $SO^{(-)}(5)$ algebra. Diagonal matrix elements of $\mathcal{H}_p$ in our basis follow easily from Eq. (4); note that $\hat{n}_2$ gives m_2 and the other part giving diagonal matrix elements is $\sum_\mu[\{A^1_\mu(j_1)[A^1_\mu(j_1)]^\dagger\} + \alpha^2\{A^1_\mu(j_2)[A^1_\mu(j_2)]^\dagger\}]$. The off-diagonal matrix elements involve $SO(5) \supset SO(3)\otimes U(1)$ reduced Wigner coefficients and using Eq. (14) of [15] will give,

$$\begin{aligned}
&\Big\langle(\omega_1^a\omega_2^a)H_f^aT_f^a\,,\;(\omega_1^b\omega_2^b)H_f^bT_f^b;T\,M_T \mid \mathcal{H}_p(\zeta,\alpha) \\
&\quad\mid (\omega_1^a\omega_2^a)H_i^aT_i^a\,,\;(\omega_1^b\omega_2^b)H_i^bT_i^b;T\,M_T\Big\rangle = -\left(\frac{4\xi}{m^2}\right)\,(\alpha) \\
&\quad\times\left[(\omega_1^a(\omega_1^a+3)+\omega_2^a(\omega_2^a+1)\right]^{1/2}\;\left[(\omega_1^b(\omega_1^b+3)+\omega_2^b(\omega_2^b+1)\right]^{1/2} \\
&\qquad\times\;(-1)^{T_f^a+T_f^b+T+1}\sqrt{(2T_f^a+1)(2T_f^b+1)}\left\{\begin{matrix} T & T_f^b & T_f^a \\ 1 & T_i^a & T_i^b\end{matrix}\right\} \\
&\qquad\quad\times\;\left\langle(\omega_1^a\omega_2^a)H_i^aT_i^a\;\;(11)1,1\;\;||\;(\omega_1^a\omega_2^a)H_f^aT_f^a\right\rangle \\
&\qquad\qquad\times\;\left\langle(\omega_1^b\omega_2^b)H_i^bT_i^b\;\;(11)-1,1\;\;||\;(\omega_1^b\omega_2^b)H_f^bT_f^b\right\rangle
\end{aligned} \tag{7}$$

for $H_f^a = H_i^a + 1$ and $H_f^b = H_i^b - 1$; $H_i^a = \frac{m_1}{2} - \Omega_1$, $H_i^b = \frac{m_2}{2} - \Omega_2$. The $\langle - - - - \,||\, - - - - \rangle$ factors above are the $SO(5) \supset SO(3)\otimes U(1)$ reduced Wigner coefficients. For the $m = 6$ system considered in the next Section, the needed Wigner coefficients follow from Tables III in [15] and Table A.1 in [16]. It is important to mention that for simplicity, in Eq. (7) we are not showing the additional label that is required as discussed in [15, 16]. This label is called β in [16]. Finally, Eq. (7) can be extended to three or more orbits by using isospin T couplings. Thus, $\mathcal{H}_p$ construction is possible with multiple $SO^{(\beta)}(5)$ algebras provided all the needed Wigner coefficients in Eq. (7) are known.

4 Applications of Multiple $SO(5)/Sp(2\Omega)$ Algebras

Electromagnetic transition operators T^{EL} and T^{ML} are one-body operators and their $SO(5) \supset [SO_T(3) \supset SO(2)] \otimes U(1)$ tensorial structure is $T^{(\omega_1,\omega_2)}_{H_1,T,M_T}$ with $(\omega_1\omega_2) = (11) \oplus (10) \oplus (00)$. More importantly, Eqs. (2) and (3) show that it is possible for T^{EL} and T^{ML} to be generators of $Sp^{(\beta)}(2\Omega)$ giving selection rules under certain conditions. We have the following results: (i) isovector parts of T^{EL} and T^{ML} will not be $Sp^{(\beta)}(2\Omega)$ scalars as the generators of this algebras are only isoscalar operators; (ii) the isoscalar part of T^{ML} with L odd (they preserve parity) or even can be $Sp^{(\beta)}(2\Omega)$ scalars provided $\beta_{j(\ell)} = (-1)^\ell$ for the $j(\ell)$ orbits; (iii) the T^{EL} with L even or odd will not be generators of any $Sp^{(\beta)}(2\Omega)$ as the $X(j_f, j_i, L)$ (see Eqs. (2 and (3)) given by the isoscalar part will not lead to a formula for β_{j_i} real. With the phase choice $\beta_{j(\ell)} = (-1)^\ell$, the selection rule from the generators that they will not change (v, t) or $(\omega_1\omega_2)$ irreps can be used in experimental tests of this phase choice. Besides this, EL and ML transitions can change seniority only by units of 2, i.e. transition for $v \to v, v \pm 2$ states are only allowed. In addition, the (m, T) dependence of say quadrupole moments and $B(E2)$'s can be written down using $SO(5)$ algebra.

In the second application, let us consider a two level system with first level having $\Omega_1 = 6$ with $-$ve parity and the second level having $\Omega_2 = 5$ with $+$ve parity. This is appropriate for nuclei in A=56-80 region so that the $(1p_{3/2}, 0f_{5/2}, 1p_{1/2})$ orbits with degenerate single particle levels give the $\Omega_1 = 6$ orbit (we will call it orbit #1 or a) and $0g_{9/2}$ gives the $\Omega_2 = 5$ orbit (we will call it orbit #2 or b). In our numerical calculations we use the system with six nucleons in the above two orbits. Then, the number of $+$ve parity basis states for $m = 6$ and $T = 0$ will be 24 as shown in Table 1. Using these basis states, the matrix for

Table 1. Basis states for the $m = 6$ system with $T = 0$ considered in Section 4. Here, $\Omega_1 = 6$ and $\Omega_2 = 5$. See text for details.

#	$\|(v_1,t_1)m_1,T_1 : (v_2,t_2)m_2,T_2;T{=}0\rangle$	#	$\|(v_1,t_1)m_1,T_1 : (v_2,t_2)m_2,T_2;T{=}0\rangle$
1	$\|(6,0),6,0 : (0,0)0,0 ; 0\rangle$	13	$\|(0,0),2,1 : (4,1)4,1 ; 0\rangle$
2	$\|(4,1),6,0 : (0,0)0,0 ; 0\rangle$	14	$\|(2,0),2,0 : (4,0)4,0 ; 0\rangle$
3	$\|(2,0),6,0 : (0,0)0,0 ; 0\rangle$	15	$\|(2,1),2,1 : (4,1)4,1 ; 0\rangle$
4	$\|(4,0),4,0 : (2,0)2,0 ; 0\rangle$	16	$\|(2,0),2,0 : (0,0)4,0 ; 0\rangle$
5	$\|(4,1),4,1 : (0,0)2,1 ; 0\rangle$	17	$\|(2,0),2,0 : (2,1)4,0 ; 0\rangle$
6	$\|(4,1),4,1 : (2,1)2,1 ; 0\rangle$	18	$\|(2,1),2,1 : (2,0)4,1 ; 0\rangle$
7	$\|(2,1),4,0 : (2,0)2,0 ; 0\rangle$	19	$\|(2,1),2,1 : (2,1)4,1 ; 0\rangle$
8	$\|(2,0),4,1 : (0,0)2,1 ; 0\rangle$	20	$\|(0,0),2,1 : (0,0)4,1 ; 0\rangle$
9	$\|(2,0),4,1 : (2,1)2,1 ; 0\rangle$	21	$\|(0,0),2,1 : (2,1)4,1 ; 0\rangle$
10	$\|(2,1),4,1 : (0,0)2,1 ; 0\rangle$	22	$\|(0,0),0,0 : (6,0)6,0 ; 0\rangle$
11	$\|(2,1),4,1 : (2,1)2,1 ; 0\rangle$	23	$\|(0,0),0,0 : (4,1)6,0 ; 0\rangle$
12	$\|(0,0),4,0 : (2,0)2,0 ; 0\rangle$	24	$\|(0,0),0,0 : (2,0)6,0 ; 0\rangle$

$\mathcal{H}_p$ defined by Eq. (6) is constructed following the formulation in Section 3. Diagonalization of $\mathcal{H}_p(\xi = 1, \alpha = \pm 1)$ will give eigenvalues that must be same as those given by Eq. (4) with $(m = 6, T = 0)$ and $(v, t) = (6, 0)$, $(4, 1)$ and $(2, 0)$. The eigenvalues are 0, $-44/m^2$ and $-84/m^2$ respectively with degeneracies 13, 9 and 2 respectively. It is easy to see that the wavefunctions are of the form $|(v_1, t_1)(v_2, t_2)(v, t)\gamma, m = 6, T = 0\rangle$ where γ are additional labels. Therefore, a sum of $\mathcal{C}_2(SO^{(a)}(5))$ and $\mathcal{C}_2(SO^{(b)}(5))$ will remove some of the degeneracies in the spectrum without changing the eigenvectors. By adding a term $-(\xi/m^2)[\mathcal{C}_2(SO^{(a)}(5)) + \mathcal{C}_2(SO^{(b)}(5))]$ to $\mathcal{H}_p(\xi, \alpha)$ we have calculated the eigenvalues for the $(m = 6, T = 0)$ system and shown in Figure 1 are the energies of the 24 states as a function of ξ for nine α values. For example, wavefunctions for the lowest two degenerate states are,

$$\begin{aligned}
|\Psi_1^{m=6,T=0}\rangle =& \sqrt{\frac{13}{35}}|4(0,0)0; 2(20)0\rangle \\
& - \alpha\sqrt{\frac{16}{35}}|2(0,0)1; 2(20)1\rangle + \sqrt{\frac{6}{35}}|0(0,0)0; 6(20)0\rangle\,, \\
|\Psi_2^{m=6,T=0}\rangle =& \sqrt{\frac{11}{42}}|6(2,0)0; 0(00)0\rangle \\
& - \alpha\sqrt{\frac{20}{42}}|4(2,0)1; 2(00)1\rangle + \sqrt{\frac{11}{42}}|2(2,0)0; 4(00)0\rangle\,.
\end{aligned} \tag{8}$$

Here the notation used is $|m_1(v_1, t_1)T_1; m_2(v_2, t_2)T_2\rangle$. Eq. (8) shows the role of α, i.e. the two $SO(5)$ algebras. As seen from Figure 1, clearly by changing (ξ, α) it is possible to study order-chaos-order transitions. Detailed analysis of this including all T's will be reported elsewhere.

Two-particle transfer strengths form the third application. As an example let us consider removal of a isovector pair from the lowest two states [these are Ψ_1 and Ψ_2 in Eq. (8)] of the $(m = 6, T = 0)$ system generating the states of $(m = 4, T = 1)$ system. To study the transfer strengths, we have diagonalized $\mathcal{H}_p(\xi = 1, \alpha = \pm 1)$ in $(m = 4, T = 1)$ space and the basis states here are 14 in number. Then, the eigenstates belong to $(v, t) = (21)$, (20) and (41) irreps in the 4 nucleon space. There are three, two and nine states respectively with these irreps and the corresponding eigenvalues are $-44/m^2$, $-40/m^2$ and 0 respectively. The transition operator for example can be chosen to be $P = [A^1_\mu(j_i)]^\dagger$ or it can be $[\mathcal{A}^1_\mu(\alpha)]^\dagger$ with $\alpha = +1$ or -1. These will not change $(v_1 t_1)$ and $(v_2 t_2)$ of the states. From Eq. (8) it is easy to see that the transfer is allowed to the two states with $(v, t) = (2, 0)$ and these are

$$\begin{aligned}
|\Phi_1^{m=4,T=1}\rangle &= \sqrt{\frac{1}{2}}|4(2,0)1; 0(00)0\rangle + \alpha\sqrt{\frac{1}{2}}|2(2,0)0; 2(00)1\rangle\,, \\
|\Phi_2^{m=4,T=1}\rangle &= \sqrt{\frac{2}{5}}|0(0,0)0; 4(20)1\rangle + \alpha\sqrt{\frac{3}{5}}|2(0,0)1; 2(20)0\rangle\,.
\end{aligned} \tag{9}$$

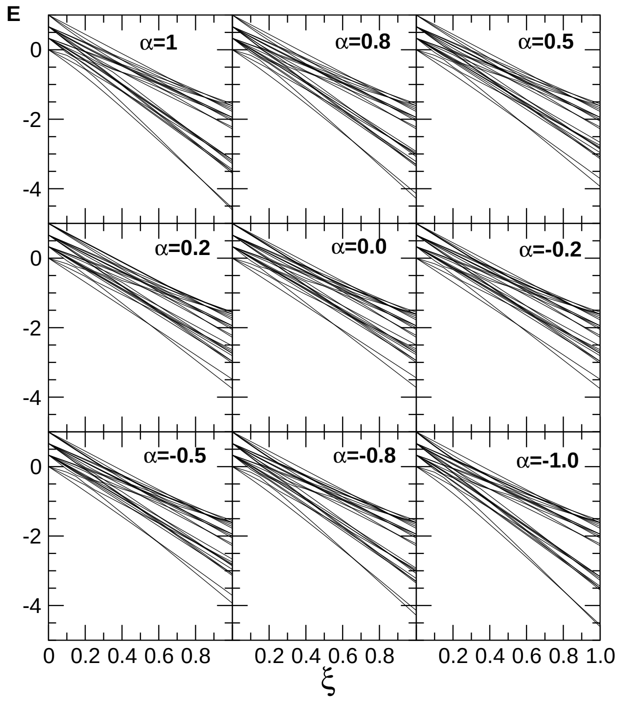

Figure 1. Energy spectra of the pairing Hamiltonian $\mathcal{H}_p$ in Eq. (6) as a function of ξ and α. Note that the energies (E) are unitless. See text for details.

With the α dependence in both the six and four particle states [see Eqs. (8) and (9)], clearly, the two-particle transfer strengths depend on α. In practice we need to add a term in the Hamiltonian that mixes the states Ψ_1 and Ψ_2 and similarly Φ_1 and Φ_2. This is being investigated and explicit formulas for the transfer strengths will be reported elsewhere.

5 Conclusions

Extending the previous results [2] on multiple $SU(2)$ pairing algebras for identical nucleons occupying several j-orbits, in this paper it is shown that there are multiple $SO(5)$ pairing algebras for nucleons (with isospin) occupying several j-orbits. Further, a method to analyze the results, based on the algebra in [15], due to multiple SO(5) (or the equivalent $Sp(2\Omega)$) algebras is described. Finally, three applications are briefly discussed. More detailed investigations of multiple $SO(5)$ algebras and their applications will be reported elsewhere. Present work complements the corresponding investigations without isospin in [2] and on multiple $SU(3)$ algebras in [34, 35].

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