

A γ -Rigid Solution of the Bohr Hamiltonian with Deformation-Dependent Mass Term for Kratzer Potential and $\gamma = 30^\circ$

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Abstract. In this work, the Davydov-Chaban Hamiltonian, describing the collective motion of γ -rigid atomic nuclei, is amended by allowing the mass parameter to depend on the nuclear deformation. Further, Z(4)-DDM (Deformation-Dependent Mass) model is proposed by considering the Kratzer potential for the β variable, and solving the problem by techniques of asymptotic iteration method (AIM). The results of the calculated spectra and $B(E2)$ transition rates for series of $^{192-196}\text{Pt}$ isotopes are compared with the corresponding experimental data as well as with other theoretical models. Exact analytical expressions are derived for spectra and normalized wave functions of the Kratzer potential. The obtained results show an overall good agreement with the experimental data and an important improvement in respect to other models.

KEY WORDS: Davydov-Chaban Hamiltonian, shape phase transition, collective models, deformation dependent effective mass, Kratzer potential.

1 Introduction

The study of shape phase transitions in nuclear structure have attracted a lot of attention from both experimental and theoretical perspectives. Therefore, several approaches have been developed in this context especially in the framework of the Bohr-Mottelson Model (BMM) [1] and the Interacting Boson Model (IBM) [2]. Furthermore, the interest devoted for this topic has increased even

more with the occurrence of Critical Point Symmetries (CPSs). Such symmetries, for example, E(5) [3] and X(5) [4] correspond to the shape phase transitions $U(5) \leftrightarrow O(6)$ and $U(5) \leftrightarrow SU(3)$, respectively. In this context, considerable attempts have been done for several potentials to achieve analytical solutions of the Bohr Hamiltonian, either in the usual case where the mass parameter is assumed to be a constant [5–9], or by introducing the Deformation-Dependent Mass Formalism (DDMF) [10–14]. In addition, another direction of research was to investigate such phenomena by imposing a γ -rigidity as in the case of Z(4) [15] or X(3) [16].

In the present work, the attention is focused on the study of the quadrupole collective states in γ -rigid case, by modifying Davydov-Chaban Hamiltonian [17] in the framework of DDMF [18–20] with Kratzer potential [21] for the variable β and γ fixed to $\pi/6$. The expressions for the energy levels as well as for the wave functions are obtained in closed analytical form by means of the Asymptotic Iteration Method (AIM) [22], an efficient technique that we have used to solve many similar problems [23–26].

The content of this paper is arranged as follows. In Section 2, the theoretical framework of Z(4)-DDM model is briefly described. In Section 3, the exact separation of the Hamiltonian and solution of angular equation are achieved. The analytical expressions for the energy levels of Kratzer potential and the wave functions are given in Section 4. Finally, Section ?? is devoted to the numerical calculations for energy spectra, $B(E2)$ transition probabilities, while Section ?? contains the main conclusions.

2 Theoretical Framework of Z(4)-DDM Model

In the framework of Davydov–Chaban model [17], the nucleus is assumed to be γ -rigid. Therefore, the Hamiltonian operator depends on four variables (β, θ_i) and has the following form [17]:

$$H = -\frac{\hbar^2}{2B} \left[\frac{1}{\beta^3} \frac{\partial}{\partial \beta} \beta^3 \frac{\partial}{\partial \beta} - \frac{1}{4\beta^2} \sum_{k=1,2,3} \frac{Q_k^2}{\sin^2(\gamma - \frac{2}{3}\pi k)} \right] + V(\beta), \quad (1)$$

where B is the mass parameter, β the collective coordinate and γ a parameter, while Q_k are the components of the total angular momentum in the intrinsic frame and θ_i the Euler angles.

In order to construct the Davydov-Chaban Hamiltonian with a mass depending on the deformation coordinate β , one follows the formalism described in Sec. II of [10] considering:

$$B(\beta) = \frac{B_0}{f(\beta)^2}, \quad (2)$$

where B_0 is a constant and $f(\beta)$ is the deformation function depending only on the radial coordinate β . Then, only the β part of the resulting equation will be

affected by deformation dependent mass. The resulting equation reads as

$$\left[-\frac{\sqrt{f}}{\beta^3} \frac{\partial}{\partial \beta} \beta^3 f \frac{\partial}{\partial \beta} \sqrt{f} + \frac{f^2}{4\beta^2} \sum_{k=1,2,3} \frac{Q_k^2}{\sin^2(\gamma - \frac{2}{3}\pi k)} \right] \Psi(\beta, \Omega) + v_{\text{eff}} \Psi(\beta, \Omega) = \epsilon \Psi(\beta, \Omega), \quad (3)$$

with

$$v_{\text{eff}} = v(\beta) + \frac{1}{4}(1 - \delta - \lambda) f \nabla^2 f + \frac{1}{2} \left(\frac{1}{2} - \delta \right) \left(\frac{1}{2} - \lambda \right) (\nabla f)^2, \quad (4)$$

where the reduced energies and potentials are defined as $\epsilon = \frac{B_0}{\hbar^2} E$ and $v(\beta) = \frac{B_0}{\hbar^2} V(\beta)$, respectively.

Considering a total wave function of the form $\Psi(\beta, \Omega) = \chi(\beta)\phi(\Omega)$, where Ω denotes the rotation Euler angles $(\theta_1, \theta_2, \theta_3)$, the separation of variables gives two equations

$$\left[-\frac{1}{2} \frac{\sqrt{f}}{\beta^3} \frac{\partial}{\partial \beta} \beta^3 f \frac{\partial}{\partial \beta} \sqrt{f} + \frac{f^2}{2\beta^2} \Lambda + \frac{1}{4}(1 - \delta - \lambda) f \nabla^2 f \right] \chi(\beta) + \frac{1}{2} \left[\left(\frac{1}{2} - \delta \right) \left(\frac{1}{2} - \lambda \right) (\nabla f)^2 + v(\beta) \right] \chi(\beta) = \epsilon \chi(\beta), \quad (5)$$

$$\left[\frac{1}{4} \sum_{k=1,2,3} \frac{Q_k^2}{\sin^2(\gamma - \frac{2}{3}\pi k)} - \Lambda \right] \phi(\Omega) = 0, \quad (6)$$

where Λ is the eigenvalue for the equation of the angular part. In the case of $\gamma = \pi/6$, the angular momentum term can be written as [27],

$$\sum_{k=1,2,3} \frac{Q_k^2}{\sin^2(\gamma - \frac{2}{3}\pi k)} = 4(Q_1^2 + Q_2^2 + Q_3^2) - 3Q_1^2. \quad (7)$$

Eq. (6) has been solved by Meyer-ter-Vehn [27], with the results

$$\Lambda = L(L+1) - \frac{3}{4}\alpha^2, \quad (8)$$

$$\phi(\Omega) = \phi_{\mu, \alpha}^L(\Omega) = \sqrt{\frac{2L+1}{16\pi^2(1+\delta_{\alpha,0})}} \left[\mathcal{D}_{\mu, \alpha}^{(L)}(\Omega) + (-1)^L \mathcal{D}_{\mu, -\alpha}^{(L)}(\Omega) \right], \quad (9)$$

where $\mathcal{D}(\Omega)$ denotes Wigner functions of the Euler angles, L is the total angular momentum quantum number, μ and α are the quantum numbers of the projections of angular momentum on the laboratory fixed z -axis and the body-fixed

x' -axis, respectively. In the literature, for the triaxial shapes, it is customary to insert the wobbling quantum number n_w instead of α , with $n_w = L - \alpha$ [27]. Within this convention, the eigenvalues of the angular part become :

$$\Lambda = L(L + 1) - \frac{3}{4}(L - n_w)^2. \quad (10)$$

3 Z(4)-DDM Solution for β Part of the Hamiltonian

The β -vibrational states of the triaxial nuclei, having a γ rigidity of $\pi/6$, are determined by the solution of the radial Schrödinger equation

$$\begin{aligned} \frac{1}{2}f^2\chi'' + \left(\frac{3f^2}{2\beta} + ff'\right)\chi' + \left(\frac{3ff'}{4\beta} + \frac{(f'^2)}{8} + \frac{ff''}{4}\right)\chi \\ - \frac{f^2}{2\beta^2}\Lambda\chi + \epsilon\chi - v_{\text{eff}}\chi = 0, \end{aligned} \quad (11)$$

with

$$v_{\text{eff}} = v(\beta) + \frac{1}{4}(1 - \delta - \lambda)ff'' + \frac{1}{2}\left(\frac{1}{2} - \delta\right)\left(\frac{1}{2} - \lambda\right)(f')^2. \quad (12)$$

By using the standard transformation of the radial wave function as $\chi(\beta) = \beta^{-3/2}R(\beta)$, one gets:

$$f^2R'' + 2ff'R' + (2\epsilon - 2u_{\text{eff}})R = 0, \quad (13)$$

with

$$u_{\text{eff}} = v_{\text{eff}} + \frac{f^2}{2\beta^2}\Lambda + \left(\frac{3ff'}{4\beta} + \frac{3f^2}{8\beta^2} - \frac{(f')^2}{8} - \frac{ff''}{4}\right). \quad (14)$$

Now, one considers the special case of the Kratzer potential type [21], defined as

$$v(\beta) = -\frac{1}{\beta} + \frac{\beta_0}{2\beta^2}, \quad (15)$$

where β_0 indicates the position of the minimum of the potential. According to the specific form of the potential (15), one chooses the deformation function in the following special form [11]:

$$f(\beta) = 1 + a\beta, \quad a \ll 1. \quad (16)$$

By inserting the potential and the deformation function in Eq. (13), one gets:

$$2u_{\text{eff}}(\beta) = k_0 + \frac{k_{-1}}{\beta} + \frac{k_{-2}}{\beta^2}, \quad (17)$$

with

$$\begin{aligned} k_0 &= \frac{a^2}{2} \left[4 + 2\Lambda + 2\left(\frac{1}{2} - \delta\right)\left(\frac{1}{2} - \lambda\right) + 3(1 - \delta - \lambda) \right], \\ k_{-1} &= a \left[3 + 2\Lambda + \frac{3}{2}(1 - \lambda - \delta) \right] - 2, \\ k_{-2} &= \frac{3}{4} + \Lambda + \beta_0. \end{aligned} \tag{18}$$

In order to apply the asymptotic iteration method [22], we propose an appropriate physical wave function for the radial function $R(\beta)$:

$$R_{n_\beta L}(\beta) = \beta^\eta (1 + a\beta)^\kappa G_{n_\beta L}(\beta), \tag{19}$$

with

$$\begin{aligned} \eta &= \frac{1}{2} (1 + \sqrt{1 + 4k_{-2}}), \\ \kappa &= -\frac{1}{2} \left(1 + \sqrt{1 + 4k_{-2} + \frac{4k_0}{a^2} - \frac{8\epsilon}{a^2} - \frac{4k_{-1}}{a}} \right). \end{aligned} \tag{20}$$

For this form, the radial wave equation reads:

$$G''(\beta) = - \left[\frac{2(\eta + a\beta(1 + \eta + \kappa))}{\beta(1 + a\beta)} \right] G'(\beta) - \left[\frac{2a\eta(\eta + \kappa) - k_{-1}}{\beta(1 + a\beta)} \right] G(\beta), \tag{21}$$

while the generalized formula of the radial energy spectrum is:

$$\epsilon_{n_\beta n_w L} = \frac{1}{2} \left[k_0 + \frac{a^2}{4} - \left(\frac{k_{-1} + an_\beta^2 + a\eta(1 + 2n_\beta)}{2(\eta + n_\beta)} \right)^2 \right], \tag{22}$$

where n_β is the principal quantum number of β vibrations.

The excited-state wave functions reads:

$$\begin{aligned} R_{n_\beta L}(\beta) &= C_{n_\beta L} a^{-\eta} 2^{n_\beta + \kappa_n} (1 + t)^\eta (1 - t)^{-\kappa_n - \eta - n_\beta} \\ &\quad \times P_{n_\beta}^{(-1 - 2(n_\beta + \kappa_n + \eta), 2\eta - 1)}(t), \end{aligned} \tag{23}$$

with $t = \frac{-1 + a\beta}{1 + a\beta}$, while

$$\begin{aligned} C_{n_\beta L} &= \left[\frac{a^{1+2\eta}}{2} \frac{(1 + 2n_\beta + 2\eta + 2\kappa_n)(1 + 2\kappa_n)}{(\eta + n_\beta)} \right]^{\frac{1}{2}} \\ &\quad \times \left[\frac{\Gamma(1 + n_\beta + 2\kappa_n)\Gamma(n_\beta + 1)}{\Gamma(n_\beta + 2\eta + 2\kappa_n)\Gamma(n_\beta + 2\eta)} \right]^{\frac{1}{2}}. \end{aligned} \tag{24}$$

The quantities k_0, k_{-1}, k_{-2} are given by Eq. (18), while Λ is the eigenvalue of the angular part given by Eq. (10). The excitation energies depend on three quantum numbers, n_β, n_w and L , respectively four parameters: a the deformation mass parameter, β_0 the minimum of the potential, the free parameters δ and λ coming from DDM formalism [28]. In the last part of the paper, a comparison to the experiment will be carried out by fitting the theoretical spectra to the experimental data. Finally, it will be shown that the predicted energy levels turn out to be independent of the choice made for δ and λ .

4 Numerical Results

The Z(4)-DDM model, presented in the previous sections, has been applied for calculating the energies of the collective states and the reduced $E2$ transition probabilities for $^{192,194,196}\text{Pt}$ isotopes. All bands (*i.e.* ground state, β and γ) are characterized by three quantum numbers n_β, n_w and L . The ground state band (g.s.) is characterized by $n_\beta = n_w = 0$, the β -band by $n_\beta = 1, n_w = 0$, and the γ -band by $n_\beta = 0, n_w = 2$ for even L levels and $n_\beta = 0, n_w = 1$ for odd L levels.

In this work, the theoretical predictions for the levels (22), are treated equally, depending on two parameters, namely: the potential minimum β_0 and the deformation dependent mass parameter a . These parameters are adjusted to reproduce the experimental data by applying a least-squares fitting procedure for each considered isotope. We evaluate the root mean square (rms) deviation between the theoretical values and the experimental data via the formula:

$$\sigma = \sqrt{\frac{\sum_{i=1}^m (E_i(\text{exp}) - E_i(\text{th}))^2}{(m-1)E(2_g^+)}}. \quad (25)$$

This quantity, represents the *rms* deviations of the theoretical calculations from the experiment, where m denotes the number of states, while $E_i(\text{exp})$ and $E_i(\text{th})$ represent the theoretical and experimental energies of the i -th level, respectively. $E(2_g^+)$ is the energy of the first excited level of the ground state band (gsb).

From Table 1, one can see that the obtained results for the levels belonging to gs, β and γ bands are in quite satisfactory agreement with experimental data. Analyzing the mean deviation corresponding for each nucleus, one can see that the present results are fairly better than those obtained by Z(4)-sextic model. This is explained by the fact that here the mass parameter depends on the β variable, while in Refs. [5, 6] the mass is considered as a constant.

Similarly, we have calculated the intraband and interband $B(E2)$ transition rates, normalized to the transition from the first excited level of the ground state band to the ground state, using the same optimal values of the three parameters

Table 1. The energy spectra comprising the ground, γ and β bands obtained with our models Z(4)-DDM Kratzer (K) are compared with the values taken from [5] and [6] with the available experimental data [29–31].

	¹⁹² Pt			¹⁹⁴ Pt			¹⁹⁶ Pt			
	Exp [29]	K	Ref. [5]	Exp [30]	K	Ref. [5]	Exp [31]	K	Ref. [5]	
R _{0,0,4}	2.479	2.451	2.439	2.470	2.506	2.415	2.465	2.455	2.513	2.481
R _{0,0,6}	4.314	4.129	3.787	4.298	4.334	3.835	4.290	4.144	3.709	3.701
R _{0,0,8}	6.377	5.844	5.773	6.392	6.306	5.880	6.333	5.877	5.579	5.559
R _{0,0,10}	8.624	7.472	7.350	8.672	8.280	7.573	8.558	7.528	6.914	6.932
R _{1,0,0}	3.776	3.768	3.397	3.858	3.806	3.706	3.192	3.124	2.954	2.977
R _{1,0,2}	4.547	4.472	4.995	4.603	4.555	5.409	3.828	3.844	4.308	4.364
R _{1,0,4}		5.506	7.002		7.511	5.693		4.904	6.238	6.280
R _{0,2,2}	1.935	1.900	1.653	1.894	1.926	1.661	1.936	1.902	1.646	1.643
R _{0,1,3}	2.910	2.714	2.302	2.809	2.786	2.332	2.852	2.719	2.249	2.252
R _{0,2,4}	3.795	4.548	4.229	3.743	4.806	4.268	3.636	4.566	4.179	4.150
R _{0,1,5}	4.682	4.748	4.342	4.563	5.034	4.402	4.526	4.769	4.243	4.227
R _{0,2,6}	5.905	6.785	6.358		7.434	6.524	5.644	6.830	6.041	6.049
R _{0,1,7}	6.677	6.637	6.065		7.255	6.235		6.681	5.737	5.754
R _{0,2,8}	8.186	8.640	9.163		9.764	-	7.730	8.717	8.564	8.573
rms		0.500	0.614		0.390	0.543		0.576	0.682	0.683
a		0.002			0.004			0.006		
β_0		50.0			70.6			51.0		

Table 2. The comparison of experimental data [29–31] (upper line) for several $B(E2)$ ratios of nuclei to predictions by the Davydov-Chaban Hamiltonian with β -dependent mass for the Kratzer potential (lower line), using the parameter values shown in Table 1.

nucleus	$\frac{4_g \rightarrow 2_g}{2_g \rightarrow 0_g}$	$\frac{6_g \rightarrow 4_g}{2_g \rightarrow 0_g}$	$\frac{8_g \rightarrow 6_g}{2_g \rightarrow 0_g}$	$\frac{10_g \rightarrow 8_g}{2_g \rightarrow 0_g}$	$\frac{2_\gamma \rightarrow 2_g}{2_1 \rightarrow 0_g}$	$\frac{2_\gamma \rightarrow 0_g}{2_g \rightarrow 0_g} \times 10^3$	$\frac{0_\beta \rightarrow 2_g}{2_g \rightarrow 0_g}$	$\frac{2_\beta \rightarrow 0_g}{2_g \rightarrow 0_g} \times 10^3$
¹⁹² Pt	1.56(12)	1.23(55)			1.91(16)	9.5(9)		
	1.58	2.38	3.32	4.61	1.60	0.0	0.70	17.36
¹⁹⁴ Pt	1.73(13)	1.36(45)	1.02(30)	0.69	1.81(25)	5.9(9)	0.01	
	1.56	2.31	3.19	4.41	1.58	0.0	0.67	25.08
¹⁹⁶ Pt	1.48(3)	1.80(23)	1.92(23)			0.4	0.07(4)	0.06(6)
	1.63	2.58	3.86	5.91	1.65	0.0	0.90	23.16
								0.3751

obtained from fitting the energy ratios. From the obtained theoretical results shown in Table 2, one can remark some discrepancies within the ground state band of the higher L levels, while the experimental values show a decreasing trend. For the interband transitions rates from the γ band to the gsb, our model give good results, while interband transitions from the β band to the gsb, the agreement is only partially good.

5 Conclusion

In this paper, based on Davydov-Chaban Hamiltonian within the framework of DDM formalism, a new model entitled Z(4)-DDM has been elaborated. The numerical realization of this model consists in calculating the energy spectra and transition probabilities of $^{192,194,196}\text{Pt}$ isotopes using Kratzer as collective potential compared to experimental data as well as with other models results. In addition, we have shown that the present model has well improved predictions in comparison with those of [5, 6].

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