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The L-Projection of the Shell Model $SU(3)$ Wave Functions

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Abstract. Elliott created the algebraic version of the Shell Model in the LS coupling scheme. He proved that one nuclear, leading wave function includes all the possible K rotational nuclear bands and states with good angular momentum L. Later on, Elliott and Harvey proposed the L-projection technique, from which one may project all the nuclear states with good L, K out of this unique leading wave function. Finally Vergados gave analytic expression for the projection coefficients, which are involved in this technique. Hereby I discuss another way to accomplish the L-projection technique, which involves the coupling of spherical tensor operators.

KEY WORDS: shell model, $SU(3)$ symmetry, interacting boson model, L-projection.

1 Introduction

The more established microscopic nuclear model is the nuclear Shell Model [1, 2]. There, it is supposed that each nucleon is orbiting in a mean field potential created by the rest of the nucleons, plus a spin-orbit term. The mean field potential is well represented by the three dimensional isotropic harmonic oscillator potential and so the spatial part of the single-nucleon orbitals can be expressed as $|n_z, n_x, n_y\rangle$, where the n_z, n_x, n_y represent the number of quanta in the the three cartesian axes [3].

Elliott was the first to apply symmetries in nuclear physics in 1958, when he introduced the Shell Model $SU(3)$ symmetry (or nowadays called the Elliott $SU(3)$ symmetry) [4, 5]. With his work Elliott explained how the nucleons in a valence shell, which consists by orbitals with common number of harmonic oscillator quanta, generate the rotational spectrum. Thus he bridged the microscopic picture given by the nuclear Shell Model of Mayer and Jensen [1, 2] with the collective and especially with the rotational nuclear properties. Elliott along with Harvey and Wilsdon had applied the Shell Model $SU(3)$ symmetry in the sd nuclear shell among the harmonic oscillator magic numbers 8-20. This work began in 1958 with Ref. [4] and lasted till 1968 with the publication of Ref. [6].

In the third article of this series [7] Elliott and Harvey proved that from a single, nuclear, leading, cartesian wave function one may project several nuclear states with good angular momentum L and projection of the angular momentum $M = K$, where K is the band label. This projection is named L-projection and was later on accomplished by Vergados in Ref. [8].

In this article I shall discuss a different technique for the L-projection. This technique is based on the fact that the leading, Shell Model $SU(3)$ wave functions consist by pairs of harmonic oscillator quanta [9]. The symmetric or antisymmetric couplings of such quanta lead to the expression of this leading state.

2 Spherical Tensor Operators

The Shell Model $SU(3)$ states are many-quanta states [9,10]. These states result from the occupancies of the Shell Model orbitals by nucleons [11]. The operators, which annihilate or create a harmonic oscillator quantum in each cartesian direction x, y, z, are the [12]:

$$a_k = \sqrt{\frac{m\omega}{2\hbar}}k + \frac{i}{\sqrt{2m\omega\hbar}}p_k, \qquad a_k^\dagger = \sqrt{\frac{m\omega}{2\hbar}}k - \frac{i}{\sqrt{2m\omega\hbar}}p_k, \tag{1}$$

with $k = x, y, z$ and p_k is the relevant momentum.

We may recall that the spherical harmonics $Y_m^{l=1}$ are (Appendix A.1 of Ref. [13]):

$$Y_{-1}^1 \propto \frac{x - iy}{\sqrt{2}}, \qquad Y_0^1 \propto z, \qquad Y_1^1 \propto -\frac{x + iy}{\sqrt{2}}. \tag{2}$$

Inspired by them, we may define a slightly different tensor operator $u_m^\dagger$ and its conjugate with components $m = -1, 0, 1$ as:

$$u_{-1}^\dagger = \frac{a_x^\dagger - ia_y^\dagger}{\sqrt{2}}, \qquad u_{-1} = \frac{a_x + ia_y}{\sqrt{2}}, \tag{3}$$

$$u_0^\dagger = a_z^\dagger, \qquad u_0 = a_z, \tag{4}$$

$$u_1^\dagger = -\frac{a_x^\dagger + ia_y^\dagger}{\sqrt{2}}, \qquad u_1 = -\frac{a_x - ia_y}{\sqrt{2}}. \tag{5}$$

Using the boson commutation relations of the $a_k^\dagger, a_k$ operators and the expression of the angular momentum in terms of these operators [12] it can be proven that the $u_m^\dagger$ operators are spherical tensors of degree 1.

This system can be inverted:

$$a_x^\dagger = \frac{u_{-1}^\dagger - u_1^\dagger}{\sqrt{2}}, \qquad a_y^\dagger = i\frac{u_{-1}^\dagger + u_1^\dagger}{\sqrt{2}}, \qquad a_z^\dagger = u_0^\dagger. \tag{6}$$

Therefore the harmonic oscillator quanta can be created/annihilated either by the traditional operators $a_k^\dagger, a_k$, or by the spherical tensors of degree 1.

Since the $u_m^\dagger$ are spherical tensors of degree 1, one may couple a pair of them, to create a spherical tensor of higher degree:
a) $L = 0$,
b) $L = 1$,
c) $L = 2$.

We may define the spherical operator $(F_M^L)^\dagger$, which creates a symmetric pair of quanta with angular momentum L and projection M (Appendix A.1 of Ref. [13]):

$$(F_M^L)^\dagger = \sum_{m,m'} (1m1m'|LM)u_m^\dagger u_{m'}^\dagger, \tag{7}$$

The spherical tensors of degree $L = 0$ are usually called s operators, while if $L = 1, 2$ the tensors are named p, d respectively. But the symmetric coupling of the quanta leads only to the s, d operators:

$$(F_0^0)^\dagger = s^\dagger, \tag{8}$$

$$(F_M^2)^\dagger = d_M^\dagger. \tag{9}$$

These operators can be used for the L-projection of the Shell Model SU(3) wave functions.

3 The Algebraic Chains in the LS Coupling Scheme

A valence nuclear shell consists of orbitals with $\mathcal{N}$ number of harmonic oscillator quanta and can be occupied by 2 protons and 2 neutrons, thus it possesses a $U(4\Omega)$ symmetry. This $U(4\Omega)$ algebra has totally antisymmetric irreps. The 4Ω dimensional space is constructed by vectors of type $|n_z, n_x, n_y, m_s, m_t\rangle$. Thus these states belong to the $U(4\Omega)$ symmetry. The orbitals $|n_z, n_x, n_y, m_s, m_t\rangle$ are occupied by nucleons, which are fermions.

Then this algebra can be decomposed as (see Fig. 7.1 of Ref. [14]):

$$U(4\Omega) = U(\Omega) \otimes U(4) \tag{10}$$

where $\Omega = \frac{(\mathcal{N}+1)(\mathcal{N}+2)}{2}$ is the number of the spatial harmonic oscillator eigenstates and 4 stands for the four possible projections of spin ($s = \frac{1}{2}$) $m_s = \pm\frac{1}{2}$

and isospin ($t = \frac{1}{2}$) $m_t = \pm\frac{1}{2}$ a nucleon may adopt.The vectors of the $U(\Omega)$ algebra are the $|n_z, n_x, n_y\rangle$ and they create an Ω dimensional space. These states are once more occupied by nucleons, which are fermions. Also the $U(\Omega)$ symmetry is solely for the spatial degrees of freedom.

Then the $U(4)$ symmetry is decomposed into the spin (S) and the isospin (T) symmetries:

$$U(4) \to SU_S(2) \otimes SU_T(2). \tag{11}$$

The spatial part of the wave function has a $U(\Omega)$ symmetry and is being decomposed as follows [4, 5, 11]:

$$U(\Omega) \supset U(3) \supset SU(3). \tag{12}$$

This decomposition leads to the calculation of the Shell Model $SU(3)$ irreps (λ, μ) as in Ref. [11]. For a certain number of particles in a given valence harmonic oscillator shell several irreps might occur. One of them is called the *leading state*, or the *highest weight state* and has been defined by Elliott in 1958 through the Eqs. (14)-(15) of Ref. [5]. The $U(3)$ symmetry is about a three dimensional space, just like the $U(\Omega)$ was about an Ω dimensional space. In the case of the $U(3)$, the three vectors are the $|n_z = 1\rangle, |n_x = 1\rangle, |n_y = 1\rangle$, which are occupied by harmonic oscillator quanta, which are bosons. Therefore infinite number of quanta may occupy a $|n_z = 1\rangle$ state in a Shell Model $SU(3)$ wave function and this would lead to an $SU(3)$ irrep $(\lambda, 0)$ with $\lambda \to \infty$.

Then the $SU(3)$ algebra is further decomposed into:

$$SU(3) \supset O(3) \supset O(2) \tag{13}$$

This decomposition involves the L-projection, in which from a single leading $SU(3)$ wave function (labeled by (λ, μ)) several physical nuclear states with good angular momentum L and projection $M = K$ are projected.

4 The Shell Model $SU(3)$ Wave Functions

The third article of the Shell Model $SU(3)$ symmetry was written by Elliott and Harvey, who wrote another article (see Ref. [10]) where he explained the details of the model. We shall now focus on Eq. (3.15) of section 3.3 of Harvey's article in Ref. [10].

There he presented that the $U(3)$ wave function is made by linear combinations of the states:

$$|pqr\rangle_{i_1,\ldots,i_{p+q+r}} = a_z^\dagger(i_1)a_z^\dagger(i_2)\ldots a_z^\dagger(i_p)a_x^\dagger(i_{p+1})\ldots a_x^\dagger(i_{p+q}) \\ a_y^\dagger(i_{p+q+1})\ldots a_y^\dagger(i_{p+q+r})\,|0\rangle\,. \tag{14}$$

The dagger operators are those introduced in Eq. (1). The labels $i_1, ..., i_{p+q+r}$ take the values $1, 2, 3, ..., A_{val}$, where A_{val} is the valence number of nucleons. It is possible that a particle number may appear more than once, or not at all. So it is possible that $i_1 = i_2 = 1$. The $a_z^\dagger(i_1)\left|0\right\rangle$ represents a quantum in the z axis from the i_1^{th} particle. Clearly in the state of Eq. (14) there are p quanta in the z axis, q quanta in the x axis and r quanta in the y axis.

Harvey wrote that the vacuum $|0\rangle$ is the state of no quanta, namely the $1s$ orbital. Since the $U(3)$ symmetry derives solely from the spatial part of the many-nucleon wave function, not from the overall, not from the spin-isospin part (see (12)), the vacuum could not be the $1s^{j=1/2}$ orbital, where $\mathbf{j} = \mathbf{l} + \mathbf{s}$ denotes the total angular momentum; the j quantum number could not be included in the vacuum state, when one is building the $U(3)$ states. The vacuum state in Eq. (14) refers only to the spatial part of the $1s^{j=1/2}$ orbital for each nucleon and in the Fock state representation is:

$$|0\rangle = |0(i_1), 0(i_2), ..., 0(i_{p+q+r})\rangle \,, \tag{15}$$

where by the $|0(i_1)\rangle$ we mean that there are no quanta due to the i_1^{th} particle etc.

The nature of the vacuum follows the nature of the vector space. For instance in the $U(4\Omega)$ level, where the states are of type $|n_z, n_x, n_y, m_s, m_t\rangle$ the vacuum is the state in which no nucleons (fermions) have occupied a $|n_z, n_x, n_y, m_s, m_t\rangle$ state. In the $U(\Omega)$ level, where the states are of the type $|n_z, n_x, n_y\rangle$, the vacuum is the state in which no nucleons (fermions) have occupied a $|n_z, n_x, n_y\rangle$ state. In the $U(3)$ level the vacuum is the state, in which no quanta (bosons) have occupied a $\left|n_k = 1\right\rangle, k = x, y, z$ state.

The action of one dagger operator on the vacuum is:

$$\begin{aligned} a_z^\dagger(i_1)\,|0(i_1), 0(i_2), ..., 0(i_{p+q+r})\rangle = \\ |1_z(i_1), 0(i_2), ..., 0(i_{p+q+r})\rangle \,, \end{aligned} \tag{16}$$

where $|1_z(i_1)\rangle$ represents 1 quantum in the z axis due to the i_1^{th} particle. Accordingly the action of two dagger operators is:

$$\begin{aligned} a_z^\dagger(i_1)a_z^\dagger(i_2)\,|0(i_1), 0(i_2), ..., 0(i_{p+q+r})\rangle = \\ |1_z(i_1), 1_z(i_2), ..., 0(i_{p+q+r})\rangle \,, \end{aligned} \tag{17}$$

and this represents 1 quantum in the z axis deriving from the i_1^{th} particle and 1 more deriving from the i_2^{th} particle.

The point is that even if $i_1 = i_2 = 1$, *i.e.*, the two i particles are the same nucleon, the $U(3)$ state is **not** constructed by the action:

$$a_z^\dagger(i_1)a_z^\dagger(i_1)\,|0(i_1)\rangle \,, \tag{18}$$

because this would result to the $|n_z, n_x, n_y\rangle = |2, 0, 0\rangle$ state, which is a state of the $\Omega = 6$ space of the $U(\Omega = 6)$ symmetry (see section 3). When one wants to produce a $U(3)$ wave function the correct action is the Eq. (17) with $i_1 = i_2 = 1$. This means that the wave function is the symmetric state of two bosons (the two quanta) coming from the same particle.

Furthermore since the states $|n_k = 1\rangle, k = x, y, z$ of the $U(3)$ symmetry are occupied by quanta (bosons), the many quanta wave function can be totally symmetric upon the interchange of the quanta. This totaly symmetric $SU(3)$ wave function would be represented by an irrep $(\lambda, 0)$. Since the quanta are bosons, infinite number of quanta can occupy one of the three states $|n_k = 1\rangle, k = x, y, z$ of the $U(3)$ symmetry.

Another example is that of a symmetric pair of quanta with $L = K = M = 0$, which can be created by the $s^\dagger$ operator when acting on the vacuum state. This operator derives from Eq. (8) and has the form:

$$s^\dagger = \frac{1}{\sqrt{3}}\left(u_1^\dagger(i)u_{-1}^\dagger(i') + u_{-1}^\dagger(i)u_1^\dagger(i') - u_0^\dagger(i)u_0^\dagger(i')\right), \tag{19}$$

where the i, i' adopt values $i_1, i_2, ..., i_{p+q+r}$. But since the $u_m^\dagger$ operators are connected with the $a_k^\dagger$ as in Eq. (6), the $s^\dagger$ can be also written as:

$$s^\dagger = -\frac{1}{\sqrt{3}}(a_x^\dagger(i)a_x^\dagger(i') + a_y^\dagger(i)a_y^\dagger(i') + a_z^\dagger(i)a_z^\dagger(i')). \tag{20}$$

5 Two Quanta in the z Axis

The easiest example is that of 1 proton or neutron in the sd nuclear shell among the nucleon magic numbers 8-20. This particle occupies the cartesian state $|n_z, n_x, n_y\rangle = |2, 0, 0\rangle$ (see Eq. (4) of Ref. [9]), thus it possesses a $U(3)$ irrep $[f_1, f_2, f_3]$= $[\sum_i n_{z,i}, \sum_i n_{x,i}, \sum_i n_{y,i}]$= $[2, 0, 0]$ (see Eq. (21) of Ref. [9]) and so the Shell Model $SU(3)$ irrep is the $(\lambda, \mu) = (2, 0)$ (see Eqs. (27), (28) of Ref. [9]). The many-quanta $SU(3)$ wave function of this irrep is represented by the quantum-number and particle-number (section 2.4.2 of Ref. [13]) Young tableaux:

$$\boxed{\mathbf{z}\,|\,\mathbf{z}} \qquad \boxed{1\,|\,2}. \tag{21}$$

In general boxes in horizontal lines in a Young tableau represent symmetric objects. Thus the quanta 1, 2 are symmetric upon their interchange, so their wave function is:

$$\Phi = a_z^\dagger(i_1)a_z^\dagger(i_2)\,|0\rangle\,. \tag{22}$$

With the use of the operators of Eq. (6), the above wave function is written as:

$$\Phi = u_0^\dagger(i_1)u_0^\dagger(i_2)\,|0\rangle\,. \tag{23}$$

The operators $u_0^\dagger(i_1)u_0^\dagger(i_2)$ can be written in terms of the $s^\dagger$, $d_0^\dagger$ operators of Eqs. (8), (9) as:

$$u_0^\dagger(i_1)u_0^\dagger(i_2) = -\frac{1}{\sqrt{3}}s^\dagger + \sqrt{\frac{2}{3}}d_0^\dagger. \tag{24}$$

So Eq. (23) is equal to:

$$\Phi = -\frac{1}{\sqrt{3}}\left(s^\dagger - \sqrt{2}d_0^\dagger\right)|0\rangle\,. \tag{25}$$

Practically this means that the cartesian wave function of Eq. (22) *projects* into a wave function with good angular momentum L, so as a nuclear state with $L = 0, K = M = 0$ (with K being the band label) is included into the cartesian wave function with probability $\frac{1}{3}$ and an $L = 2, K = M = 0$ nuclear state lies within the cartesian wave function with probability $\frac{2}{3}$.

This procedure is called *L-projection* and was introduced by J. P. Elliott in 1958 in Ref. [5]. The projection operator P is further explained in the Appendix of Ref. [7], while the matrix elements of P have been calculated in 1968 by J. D. Vergados in Ref. [8]. The matrix elements of the projection operator within the same K nuclear band are:

$$A((\lambda,\mu),K,L,K) = \langle\Phi|P|\Phi\rangle = |a((\lambda,\mu),K,L)|^2. \tag{26}$$

The coefficients $a((\lambda,\mu),K,L)$ are given in Table 2A of Ref. [8]. The $SU(3)$ irrep of our example is the (λ,μ)= $(2,0)$ and so:

$$a((2,0),K=0,L=0) = \frac{1}{\sqrt{3}}, \tag{27}$$

$$a((2,0),K=0,L=2) = \sqrt{\frac{2}{3}}. \tag{28}$$

One way or another the cartesian state of two quanta in the z cartesian axis corresponds to an s pair of spherical quanta with probability $\frac{1}{3}$ and to a d pair with probability $\frac{2}{3}$.

Eq. (25) can be written as:

$$\Phi = \frac{1}{1+\beta^2}\left(-s^\dagger + \beta\cos\gamma d_0^\dagger + \frac{1}{\sqrt{2}}\beta\sin\gamma(d_2^\dagger + d_{-2}^\dagger)\right)|0\rangle \tag{29}$$

with $\beta = \sqrt{2}$ and $\gamma = 0°$. In the above equation the minus in the $s^\dagger$ operator is due to the minus in the overall coefficient in Eq. (20), *i.e.*, the $-s^\dagger$ is positive. Thus, it looks quite similar with the coherent state of Ginocchio and Kirson [15], with the difference that in the coherent states the $s^\dagger, d_M^\dagger$ operators are the boson creation operators of the Interacting Boson Model [16].

6 The Difference with the Symplectic Model

This idea can be easily confused with the Symplectic Model of Rowe and Rosensteel [17]. In Eq. (5a) of Ref. [17] the authors treat the operator:

$$A_{ij} = \frac{1}{2} \sum_{\alpha} c^{\dagger}_{\alpha i} c^{\dagger}_{\alpha j}, \tag{30}$$

where the $c^{\dagger}_{\alpha i}$ is the $a^{\dagger}_{k}(i)$ operator of the present manuscript. The operator A_{ij} has the B_{ij} as a conjugate and their commutation relation is given in Eq. (6) of Ref. [18]. This commutation relation indicates that these operators are not exactly boson operators.

In the Symplectic Model the A_{ij} operator is acting upon the $|n_z, n_x, n_y, m_s, m_t\rangle$ state, which is occupied by the α^{th} nucleon (fermion). This action causes a particle excitation of the fermion across two major harmonic oscillator shells. Thus the A_{ij} is acting on the states, which possess both the spatial and the spin-isospin degree of freedom. Such states are being occupied by fermions, so correctly the commutator of Eq. (6) of Ref. [18] outlines that the A_{ij} and the B_{ij} do not obey exactly the boson statistics.

On the contrary, in the Shell Model SU(3) wave function in Harvey's review [10] the action of the $a^{\dagger}_{k}(i)a^{\dagger}_{k'}(i')$ on the relevant vacuum places two harmonic oscillator quanta (bosons) of the *valence* harmonic oscillator shell in the $|n_k = 1\rangle$, $|n_{k'} = 1\rangle$ (with $k, k' = x, y, z$) states. Since the quanta are bosons the overall many-quanta wave function can be symmetric upon the interchange of the two quanta. And this is how we result to have symmetric pairs of quanta and the $s^{\dagger}$ and $d^{\dagger}_{M}$ operators, when we are doing the L-projection. So in the $U(3)$ wave functions a state $|n_k = 1\rangle$ can be occupied by infinite number of quanta and all the quanta belong to the valence shell, *i.e.*, there are no particle excitations.

So despite that the operator A_{ij} looks very similar with the $a^{\dagger}_{k}(i)a^{\dagger}_{k'}(i')$ of the present idea, there are substantial differences among the two approaches:

a) in the Symplectic Model the operators are acting upon the states, which are characterized by both the spatial and the spin-isospin degrees of freedom, while the Shell Model $U(3)$ states are solely about the spatial degrees of freedom,
b) the states in the Symplectic Model are occupied by fermions (the nucleons), while the states of the Shell Model $U(3)$ are occupied by bosons (the quanta),
c) the actions of the ladder operator in the Symplectic Model results to particle excitations across two major harmonic oscillator shells, while the action of the ladder operators in the Shell Model $SU(3)$ is about the symmetric state of two quanta of the valence shell.

7 Conclusions

Another L-projection technique for the Shell Model $SU(3)$ wave functions was described. This procedure involves the coupling of the harmonic oscillator quanta, which are the building blocks of the leading, nuclear wave functions in the Shell Model $SU(3)$ symmetry. A spherical tensor operator of degree 1 was introduced, which creates a harmonic oscillator quantum with angular momentum $l = 1$ and projection m. The symmetric coupling of two harmonic oscillator quanta leads to the operators s, d, which are spherical tensors of degree 0, 2 respectively. A very simple leading wave function has been described, the one of two quanta in the z axis. This leading state was projected into states with good L, K, with the use of the s, d operators. Surprisingly this leading state, when expressed in terms of the s, d operators has the form of a coherent state of Ginnocchio and Kirson, if the s, d operators were the s, d bosons of the Interacting Boson Model. So the question is “can the s, d bosons in the $SU(3)$ limit of the Interacting Boson Model be interpreted as symmetric pairs of harmonic oscillator quanta?”

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