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Dark Energy Bianchi Type-III Cosmological Models in $f(T)$ Theory of Gravity

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Abstract. In this paper, we have considered spatially homogeneous and anisotropic Bianchi type-III universe in the context of $f(T)$ theory of gravity. Here, we have reconstructed Bianchi type-III cosmological models using continuity equation and equation of state parameter which represent different phases of universe. We have considered matter dominated era, radiation dominated era and dark energy phase along with their combinations. It has been observed that one of the models has constant solution which may correspond to cosmological constant. We have also derived equation of state parameter by using well known $f(T)$ models and described cosmic acceleration.

KEY WORDS: $f(T)$ gravity, Bianchi typeIII universe, continuity equation.

1 Introduction

The concept of accelerated expansion of universe put forward by Supernova type-Ia experiment has revolutionized modern cosmology [1, 2]. There are two representative approaches to explain this acceleration. The first is that universe is fluid by an exotic fluid with negative pressure called Dark Energy (DE) generally materialized by the cosmological constant within General Relativity (GR). The second is modifying the gravitational action and so, explanations can be done about the acceleration of the expansion of the universe. There have been several investigations on this way within theories essential based on the curvature scalar [3–6].

One of the most popular modified gravity theory which is obtained by modifying Einstein–Hilbert (EH) action by replacing Ricci curvature scalar R with an $f(R)$ function, which is an arbitrary function of R, is called $F(R)$ theory of gravity [7–12]. It was observed that the late time acceleration of universe can be explained within this modified theory [13]. Interesting result are obtained with

$f(R)$ gravity [14–16] but its equations are of order four so that mathematical difficulties are more acute in this theory.

Harko *et al*. [17] developed another modified theory of gravity known as $f(R, T)$gravity which is generalization of $f(R)$ gravity. Here the Lagrangian includes a function of the scalar curvature R and the trace of energy momentum tensor T. Ferraro & Fiorini [18, 19] have presented a new modified $f(T)$theory and solved practical horizontal problem as well as obtained singularity free solutions with positive cosmological constant. $f(T)$theory of gravity is based on modified teleparallel gravity in which the torsion will be responsible for late-time accelerated expansion and the second order fields equations and it presents encouraging results [20–25]. Said *et al.* [26] examined the violation of the equivalence principle in the electromagnetic sector and its consequences in $f(T)$ gravity. Recently some authors studied the Bianchi identities in $f(T)$ gravity [27].

Spatially homogeneous and anisotropic cosmological models play a significant role in the description of large scale behaviour of the universe and such models have been widely studied in framework of general relativity in search of a realistic picture of the universe in its early stage. Though Bianchi type-I Universe is the prime candidate for studying the possible effects of an anisotropy in the early universe on present day observations, there are few other models like Bianchi type-II, III, V, VI_0, VIII and IX which describe anisotropic space-time which generate particular interest among physicists. Adhav *et al*. [28] explored Bianchi type-III cosmological model with negative constant deceleration parameter in Bran Dicke (BD) scalar tensor theory of gravitation. Adhav *et al*. [29] have studied Bianchi type-III magnetized wet dark fluid cosmological model in general relativity. Many authors studied Bianchi Type models in the context of BD scalar tensor theory of gravitation [30–32].

In this paper, we study Bianchi type-III universe in $f(T)$theory of gravity. We briefly review formalism of $f(T)$gravity in Section 2. In Section 3, we developed the field equations for Bianchi type-III universe. Section 4 is related to construction of $f(T)$ gravity models by using continuity equation. Summary and concluding remarks are given in the last section.

2 F(T) Gravity Formalism

$f(T)$ theories are based on modified teleparallel gravity. The teleparallel approach was considered originally by Einstein *et al.* [33] by using the vierbein as dynamical object. In teleparallel gravity the action is constructed by teleparallel Lagrangian $L_T = T$ which is the idea of $f(T)$ gravity is to generalize T to a function $T + f(T)$, which is similar in spirit to the generalization of Ricci scalar Rin the Einstein–Hilbert action to a function $f(R)$. In particular the action of

$f(T)$ gravity defined as

$$I = \frac{1}{16\pi G} \int d^4x \, e \left[T + f\left(T\right) + L_m\right], \tag{1}$$

where $e = \sqrt{-g}$ and L_m stands for the matter Lagrangian. G is the gravitational constant, T is the torsion scalar, $f(T)$ is a differentiable function of torsion scalar T.

One can define the teleparallel Lagrangian as

$$T = S_\rho^{\mu\upsilon} T_{\mu\upsilon}^\rho \tag{2}$$

where

$$S_\rho^{\mu\upsilon} = \frac{1}{2}\left(K_\rho^{\mu\upsilon} + \delta_\rho^\mu T_\theta^{\theta\upsilon} - \delta_\rho^\upsilon T_\theta^{\theta\mu}\right) . \tag{3}$$

Moreover, the contorsion tensor which is equal to the difference between Weitzenböck and Levi-Civita connection is define as

$$K_\rho^{\mu\upsilon} = -\frac{1}{2}\left(T_\rho^{\mu\upsilon} - T_\rho^{\upsilon\mu} - T_\rho^{\mu\upsilon}\right) , \tag{4}$$

$S_\rho^{\mu\upsilon}$ is the antisymmetric tensor.

Differing from General Relativity Teleparallelism use the curvatureless Weitzenböck connection, whose non-null torsion is

$$T_{\mu\upsilon}^\rho = \Gamma_{\mu\upsilon}^\rho - \Gamma_{\upsilon\mu}^\rho = h_i^\rho\left(\partial_\mu h_\upsilon^i - \partial_\upsilon h_\mu^i\right) , \tag{5}$$

where $\Gamma_{\upsilon\mu}^\rho$ is the Weitzenböck connection.

The metric tensor is obtained from dual vierbein as

$$g_{\mu\upsilon} = \eta_{ij} h_\mu^i h_\upsilon^i , \quad i, j, \ldots = 0, 1, 2, 3; \;\; \mu, \upsilon, \ldots = 0, 1, 2, 3 , \tag{6}$$

where the vierbein field $h_i\left(x^\mu\right)$ which is an orthonormal basis for the tangent space at each point x^μ of manifold: $e_i e_j = n_{ij}$, where $n_{ij} = \text{diag}\left(1, -1, -1, -1\right)$ is the Minkowski metric.

We now put cosmological framework equation controlled by $f(T)$ gravity. The equations of motion are given by variation of action (1) with respect to the vierbein

$$\left[e^{-1}\partial_\mu\left(eS_i^{\mu\upsilon}\right) - h_i^\lambda T_{\mu\lambda}^\rho S_\rho^{\upsilon\mu}\right] f_T + S_i^{\mu\upsilon}\partial_\mu\left(T\right) f_{TT} + \frac{1}{4} h_i^\upsilon f = \frac{1}{2}k^2 h_i^\rho T_\rho^\upsilon , \tag{7}$$

where $k^2 = 8\pi G$, $f_T = \dfrac{df}{dT}$, $f_{TT} = \dfrac{d^2 f}{dT^2}$, $S_i^\mu = h_i^\rho S_\rho^{\mu\upsilon}$.

The matter energy momentum tensor T_ρ^υ has the form

$$T_\rho^\upsilon = \text{diag}\left(\rho_m, -p_m, -p_m, -p_m\right) \quad , \tag{8}$$

where ρ_m and p_m stands respectively for energy density and pressure of the matter content of the universe.

3 The Field Equations

The spatially homogeneous and anisotropic Bianchi type-III model is given by

$$ds^2 = dt^2 - A^2\,(t)\,dx^2 - B^2 e^{-2\alpha x} dy^2 - C^2 dz^2\,, \tag{9}$$

where α is a constant parameter and $A\,(t)$, $B\,(t)$ and $C\,(t)$ are the cosmic scale factors.

We get the following set of diagonal tetrads from equation (4) and equation (9) as:

$$h^i_\mu = \text{diag}\left(1, A, Be^{-\alpha x}, C\right), h^\mu_i = \text{diag}\left(1, A^{-1}, B^{-1}e^{\alpha x}, C^{-1}\right)\,. \tag{10}$$

By using equations (2), (3), (4) and (5), the torsion tensor for Bianchi type-III is given by

$$T = -2\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC}\right)\,. \tag{11}$$

By using equations (8)–(11), for $i = 0 = \upsilon$ and $i = 1 = \upsilon$, in field equations (7) of $f(T)$ theory of gravity, we obtain the following equations:

$$f - 4\Big(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{\alpha^2}{2A^2}\Big)f_T = 2k^2\rho_m\,, \tag{12}$$

$$\begin{aligned}
&2\Big(\frac{\dot{A}\dot{B}}{AB} + 2\frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\Big)f_T \\
&- 4\Big(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Big)\Bigg[\Big(\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2}\Big)\Big(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Big) + \Big(\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2}\Big)\Big(\frac{\dot{C}}{C} + \frac{\dot{A}}{A}\Big) \\
&+ \Big(\frac{\ddot{C}}{C} - \frac{\dot{C}^2}{C^2}\Big)\Big(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\Big)\Bigg]f_{TT} - f = 2k^2 p_m\,.
\end{aligned} \tag{13}$$

The conservation equation takes the form

$$\dot{\rho}_m + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)(\rho_m + p_m) = 0\,. \tag{14}$$

Along the direction of x, y and z axes respectively, The Hubble parameter H_i are given by

$$H_1 = \frac{\dot{A}}{A}, \quad H_2 = \frac{\dot{B}}{B}, \quad H_3 = \frac{\dot{C}}{C}\,. \tag{15}$$

For Bianchi type-III, the corresponding average scale factor R, the Hubble parameter H and the anisotropy parameter Δ will become

$$R = \left(ABCe^{-\alpha x}\right)^{1/3}, \tag{16}$$

$$H = \frac{1}{3}\Big(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Big), \tag{17}$$

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\Big(\frac{H_i - H}{H}\Big)^2. \tag{18}$$

For $\Delta = 0$, it is noticed that isotropic behavior of the universe is obtained, which depends on the values of unknown scale factors and parameters that are include in the model [34–36].

From equation (11) and equation (17), we get

$$T = -9H^2 + J \tag{19}$$

with $J = \frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}$, which can be rewritten as

$$H = \frac{1}{3}\sqrt{J - T}\,. \tag{20}$$

For $f\,(T) = T$, $f_T = 1$, in equations (12) and (13), takes the form

$$\rho_m + \rho_T = \frac{1}{2k^2}\Big[-4\Big(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{\alpha^2}{2A^2}\Big) + T\Big], \tag{21}$$

$$p_m + p_T = \frac{1}{2k^2}\Big[2\Big(\frac{\dot{A}\dot{B}}{AB} + 2\frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\Big) - T\Big], \tag{22}$$

where ρ_T and p_T denotes torsion contribution to the energy density and pressure given by

$$\rho_T = \frac{1}{2k^2}\Big[-4\Big(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{\alpha^2}{2A^2}\Big)(1 - f_T) + T - f\Big], \tag{23}$$

$$\begin{aligned} p_T = \frac{1}{2k^2}\Bigg[&2\Big(\frac{\dot{A}\dot{B}}{AB} + 2\frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\Big)(1 - f_T) \\ &+ 4\Big(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Big)\Big[\Big(\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2}\Big)\Big(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Big) + \Big(\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2}\Big)\Big(\frac{\dot{C}}{C} + \frac{\dot{A}}{A}\Big) \\ &+ \Big(\frac{\ddot{C}}{C} - \frac{\dot{C}^2}{C^2}\Big)\Big(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\Big)\Big] f_{TT} - T + f\Bigg]. \end{aligned} \tag{24}$$

The following solution can easily obtained by putting $\rho_m = 0$, i.e. the homogeneous part of equation (12)

$$f\,(T) = \frac{c_0}{\sqrt{(T + G)}} \quad \text{with} \quad G = \frac{\alpha^2}{A^2}, \tag{25}$$

where c_0 is the constant of integration. By using above equation (25), equation (13) reduces to

$$p_m = \frac{1}{2k^2}\left[\frac{3M\dot{T}}{2(T+G)^2} - \frac{3\dot{H}+J+L}{(T+G)} - \frac{T+2G}{2(T+G)}\right]\frac{c_0}{\sqrt{T}}\,, \tag{26}$$

where $L = \dfrac{\dot{B}\dot{C}}{BC} - \frac{\ddot{A}}{A}$, $M = \dfrac{\dot{B}}{B} + \frac{\dot{C}}{C}$.

4 Construction of Some $f(T)$ Models Using Continuity Equation

Here, for different cases of perfect fluid, we construct some $f(T)$ model by using the continuity equation (14) and then we discussed the values of EoS parameter ω and its different combinations for non-relativistic matter, radiation and DE.

Now we consider the relation [37] for Bianchi type-III universe

$$\frac{1}{9}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)^2 = H_0^2 + \frac{k^2\rho_0}{3ABCe^{-\alpha x}}\,, \tag{27}$$

where ρ_0 is an integration constant and H_0 is the Hubble constant.

It gives

$$\left(ABCe^{-\alpha x}\right)^{-1} = \frac{3}{k^2\rho_0}\left(H^2 - H_0^2\right)\,. \tag{28}$$

Using EoS parameter, equation (14) takes the form

$$\frac{\dot{\rho}_m}{\rho_m} + 3H\left(1+\omega\right) = 0\,. \tag{29}$$

Now we construct the $f(T)$ models by using different cases of fluids and their combinations such that $\omega = 0$ for non-relativistic, $\omega = 1/3$ for relativistic matter and for DE era it is $\omega = -1$ [38].

Case-1

For non-relativistic matter, i.e., $\omega = 0$. It looks like cold dark matter (CDM) and baryons. We put $\omega = 0$ in equation (29) and using equation (28), the equation (29) reduces to

$$\rho_m = \rho_c\left(ABCe^{-\alpha x}\right)^{-1} = \frac{3\rho_c}{k^2\rho_0}\left(H^2 - H_0^2\right)\,, \tag{30}$$

where ρ_c is a constant of integration.

Also, the above equation in terms of torsion scalar can be written as

$$\rho_m = \frac{\rho_c}{3k^2\rho_0}\left(J - 9H_0^2 - T\right)\,. \tag{31}$$

Using equation (31), equation (12) becomes

$$2\,(T+G)\,f_T + f = \frac{2\rho_c}{3\rho_0}\left(J - 9H_0^2 - T\right)\,. \tag{32}$$

It gives the solution as

$$f\,(T) = \frac{\rho_c}{3\rho_0}\left(\frac{1}{\sqrt{T+G}}\int\frac{1}{\sqrt{T+G}}\left[J - 9H_0^2 - T\right]dT\right). \tag{33}$$

If the value of scale factor J is known then above equation gives unique solution. Hence for Matter dominated era, we obtain a model in terms of Torsion scalar T and Hubble constant H_0.

Case-2

For relativistic matter, i.e. $\omega = 1/3$, it is like photons and massless neutrinos which shows the radiation dominated era of the universe.

Substituting $\omega = 1/3$ in equation (29) and by using equation (19) and equation (28), equation (29) takes the form

$$\rho_m = \frac{\rho_r}{3^{4/3}k^{8/3}\rho_0^{4/3}}\left(J - 9H_0^2 - T\right)^{4/3}\,, \tag{34}$$

where ρ_r is another constant of integration.

Using above values of ρ_m in equation (12), it implies

$$2\,(T+G)\,f_T + f = \frac{2\rho_r}{3^{4/3}k^{2/3}\rho_0^{4/3}}\left(J - 9H_0^2 - T\right)^{4/3}\,. \tag{35}$$

The solution of equation (35) is

$$f\,(T) = \frac{\rho_r}{3^{4/3}k^{2/3}\rho_0^{4/3}\sqrt{T+G}}\left(\int\left[\frac{1}{\sqrt{T+G}}(J - 9H_0^2 - T)\right]^{4/3}dT\right). \tag{36}$$

Also, for relativistic matter, the solution of above equation (36) depends upon the values of J, torsion scalar T and Hubble constant H_0.

Case-3

For DE era, i.e. $\omega = -1$. DE is highly dominated component of the universe.

For $\omega = -1$, equation (29) takes the form

$$\rho_m = \rho_d\,, \tag{37}$$

Since ρ_d is the constant of integration.

By using equation (37), equation (12) reduces to

$$2\,(T+G)\,f_T + f = 2k^2\rho_d\,. \tag{38}$$

Equation (38) gives the following solution:

$$f\,(T) = 2k^2\rho_d\,. \tag{39}$$

This solution corresponds to the constant model which itself consistent with cosmological constant.

Case-4

Combination of Dust fluid and radiation, i.e. $\omega = 0$ and $\omega = 1/3$. Here we take the combination of two different fluids, the dust fluid and the radiations.

Addition of equation (31) and equation (34) gives

$$\rho_m = \frac{1}{3k^2\rho_0}(J - 9H_0^2 - T)\Big[\rho_c + \frac{\rho_r}{3^{1/3}k^{2/3}\rho_0^{1/3}}\left(J - 9H_0^2 - T\right)^{1/3}\Big]. \tag{40}$$

Putting the value of ρ_m in equation (12), we have

$$\begin{aligned} 2(T+G)f_T + f = \frac{2}{3\rho_0}(J - 9H_0^2 - T) \\ \times \left[\rho_c + \frac{\rho_r}{3^{1/3}k^{2/3}\rho_0^{1/3}}(J - 9H_0^2 - T)^{1/3}\right]. \end{aligned} \tag{41}$$

It implies the solution

$$\begin{aligned} f(T) = \frac{\rho_c}{3\rho_0}\Big(\frac{1}{\sqrt{T+G}}\int\frac{1}{\sqrt{T+G}}[J - 9H_0^2 - T]dT\Big) \\ + \frac{\rho_r}{3^{4/3}k^{2/3}\rho_0^{4/3}\sqrt{T+G}}\left(\int\left[\frac{1}{\sqrt{T+G}}[J - 9H_0^2 - T]\right]^{4/3}dT\right). \end{aligned} \tag{42}$$

Case-5

Combination of Dust fluid and DE, i.e. $\omega = 0$ and $\omega = -1$, which gives

$$\rho_m = \frac{\rho_c}{3k^2\rho_0}\left(J - 9H_0^2 - T\right) + \rho_d\,. \tag{43}$$

From equation (43), equation (12) becomes

$$2\,(T+G)\,f_T + f = \frac{2\rho_c}{3\rho_0}\left(J - 9H_0^2 - T\right) + 2k^2\rho_d\,. \tag{44}$$

We obtain the following solution:

$$f(T) = \frac{\rho_c}{3\rho_0}\left(\frac{1}{\sqrt{T+G}}\int\frac{1}{\sqrt{T+G}}\left[J - 9H_0^2 - T\right]dT\right) + 2k^2\rho_d\,. \tag{45}$$

Case-6

Combination of DE and Radiation, i.e. $\omega = -1$ and $\omega = 1/3$. We consider the combination of corresponding EoS parameter of DE and radiation dominated era which yields

$$\rho_m = \rho_d + \frac{\rho_r}{3^{4/3}k^{8/3}\rho_0^{4/3}}\left(J - 9H_0^2 - T\right)^{4/3} . \quad (46)$$

Inserting the value of ρ_m in equation (12), gives

$$2(T+G)f_T + f = 2k^2\rho_d + \frac{2\rho_r}{3^{4/3}k^{2/3}\rho_0^{4/3}}(J - 9H_0^2 - T)^{4/3} . \quad (47)$$

From equation (47) we get the solution

$$f(T) = 2k^2\rho_d + \frac{\rho_r}{3^{4/3}k^{2/3}\rho_0^{4/3}\sqrt{(T+G)}}\int \frac{1}{\sqrt{(T+G)}}(J - 9H_0^2 - T)^{4/3}dT . \quad (48)$$

It is worthwhile to point out here that the cases 4-6 provide $f(T)$ models for combinations of different matter. In fact, we develop the theory of dark matter and dark energy independently. But several attempts [39] were made which involve the interaction of different matters, so that their combined effect can be studied. Dark matter plays a vital role in the evolution of galaxy; moreover it has measurable effect on anisotropies. These cases may provide an interaction between dark matter and dark energy and drive transition from early matter dominated era to a phase of accelerated expansion.

5 Conclusions

We have studied Bianchi type-III cosmological models in $f(T)$ theory of gravity (by reconstruction of these models) using continuity equation. These models represent different phases of the universe such as matter ($\omega = 0$), radiation ($\omega = 1/3$) and DE ($\omega = -1$). Matter dominated era represents expansion of the universe filled with non-interacting dust particles while radiation dominated era explains early universe filled with radiation. The DE era corresponds to the universe filled with an exotic fluid with negative pressure causing late-time acceleration.

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