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Analytical Expressions to Energy Eigenvalues of the Hydrogenic Atoms and the Heavy Light Mesons in the Framework of 3D-NCPS Symmetries Using the Generalized Bopp's Shift Method

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Abstract. In the present paper, within the framework of the Deformed Schrödinger equation, using the generalized Bopp's shift method and standard perturbation theory, we have obtained the energy eigenvalues of a newly proposed improved trigonometric Rosen-Morse potential model (ITRMP) for the hydrogenic atoms and the Heavy Light Mesons (HLM) $Q\overline{Q}(Q = b, c)$ and we also applied the results obtained in the production of the modified mass-spectra of the HLM in the three-dimensional nonrelativistic noncommutative phase space (3D-NCPS) symmetries. The potential is a superposition of the trigonometric Rosen-Morse potential and new central terms appear as a result of the effects of noncommutativity properties of space and phase on the TRMP model. The obtained energy eigenvalues and modified mass spectra appeared as a function of the discreet atomic quantum numbers (j, n, l, s and m), infinitesimal parameters (Θ, λ and τ) and (η, $\overline{\lambda}$ and $\overline{\tau}$) which are induced by (space-space) and (phase-phase), in addition to, the dimensional parameters (μ, b, d) of the studied potential. Furthermore, we have shown that the corresponding new Hamiltonian operator is the sum of the ordinary Hamiltonian operator of the TRMP model and three operators, the first one is the modified spin-orbit interaction, the second is the modified Zeeman operator while the third part is the rotational part for the previous hydrogenic and the HLM systems.

KEY WORDS: Schrödinger equation, trigonometric Rosen-Morse potential, star product, noncommutative geometry; generalized Bopp's shift method.

1 Introduction

1.1 Averaging

Since the emergence of the nonrelativistic Schrödinger equation SE and its applications have expanded to include more fields. This equation describes the time-dependent states of low-energy quantum systems. The search for solutions to this equation using different potentials enables us to shed light on all the physical information. In recent years, there has been an intense activity of researchers through the use of the Schrödinger equation to find the energy spectrum of the quarkonium system. Badalyan investigated a phenomenological potential of quark-antiquark interaction $V(R) = -\alpha_s/R + \beta R$ within the quasiclassical framework and show that the Van Royen-Weisskopf relation has a different origin for pseudoscalar and vector mesons [1]. Kumar and Chand (2013) solved the N-dimensional radial Schrödinger equation RSE for the quark-antiquark interaction potential $U(r) = ar^2 + br - c/r$ and employed asymptotic interaction method via an Ansatz to the wave function and obtained energy spectra of the consigned system and derived the mass spectra of heavy quarkonia in 3D [2]. Abu-Shady studied heavy quarkonia and mesons in the Cornell potential CP with Harmonic oscillator potential in the ND-SE [3]. By using the parametric Nikiforov-Uvarov NU and asymptotic iteration method Ikhdair and Babatunde studied the approximate analytic bound state eigensolutions of the RSE for the Hellmann potential [4]. Hamzavi studied the same potential using the generalized parametric NU method [5]. In 2017, Onate *et al.*used the supersymmetric approach and studied the approximate analytic solutions of the 3D-SE with the Hellmann potential by applying a suitable approximation scheme to the centrifugal term [6]. Kher *et al.* studied the mass spectra of the B and B_S mesons using a CP incorporated with a $O(1/m)$ correction in the potential energy term and expansion of the kinetic energy term up to $O(p10)$ for relativistic correction of the Hamiltonian using a Gaussian wave function [7]. In 2018, the heavy-light mesons HLM in the NR quark model have been investigated by Abu-Shady *et al.* using the Laplace transformation method [8]. Ibekwe *et al*. solved the RSE with an exponential, generalized, anharmonic CP using the series expansion method [9]. Thermodynamic properties of HLM are calculated within the framework of the ND-RSE by Abu-Shady *et al*. under extended CP [10]. Al-Oun *et al.* examined characteristic heavy quarkonia $Q\overline{Q}$ $(Q = b, c)$ properties in the general potential model consisting of a Coulomb and quadratic potential in the case of SE [11]. In the same context, Kuchin *et al*. obtained the spin-averaged mass spectra of heavy quarkonia under the CP in the framework of the RSE [12]. Rahmani *et al.* solved the SE with a potential containing Coulomb, linear and quadratic terms, using the NU method [13]. Moazami *et al*. calculated the mass spectrum and decay properties of HLM under a new potential combination containing Cornell, Gaussian and inverse square terms in the NR regime [14]. Sun and Saad evaluated Shannon entropy for the position and

momentum eigenstates of an asymmetric trigonometric Rosen-Morse potential TRMP for the ground and first excited states and calculated the position and momentum information entropies numerically [15]. Compean and Kirchbach have shown that the exact solution of the SE with the TRMP in a flat three-dimensional Euclidean space was exactly Fourier transformable to momentum space [16]. Najafizade *et al.* investigated the information-theoretic measures of (1+1)-dimensional Dirac equation in both position and momentum spaces for the TRMP [17]. Deta *et al.* investigated analytically using the NU method and obtained the energy spectra and wave Function of SE in D-Dimensions for TRMP as an effective quantum chromodynamics potential [18]. Jasso and Kirchbach explain that the TRMP can be a candidate to describe the quark physics in so far as it captures the essentials of the QCD quark-gluon dynamics describe, in addition, this potential interpolates between a Coulomb-like potential and the infinite wall potential and reproduces in the intermediary region the linear confinement potential as established by lattice QCD calculations of hadron properties [19]. Abu-Shady and Ezz-Alarab solved analytically the NRSE using an exact-analytical iteration method and obtained the energy eigenvalues and corresponding wave functions and calculating the mass of mesons charmonium $c\bar{c}$, bottomonium $b\bar{b}$, $b\bar{c}$ and $c\bar{s}$ mesons and thermodynamic properties the mean internal energy, the specific heat, the free energy, and the entropy which has the following form [20]:

$$V_{\text{trm}}(r) = \frac{1}{2\mu d}\left(-2b\cot\left(\frac{r}{d}\right) + \frac{a(a+1)}{\sin^2(r/d)}\right), \tag{1}$$

where $a = 1, 2, 3$ and μ, b, d are the parameters of the potential.

1.2 Motivation

Relativistic and nonrelativistic quantum mechanics QM has achieved upper successes in terms of the convergence of theoretical treatments with experimental measurements. However, until now, some data and indications remain that point strongly to unresolved cases based on QM for example the non-renormalizable of electroweak interactions, the non-regularization of quantum field theories, quantum gravity, string theory. The idea of noncommutativity resulting from properties of deformation of space-space (Heisenberg in 1930 is the first to suggest the idea and then it was developed by Snyder in 1947) was one of the major solutions to these problems, as a result of all these motivational data, it is logical to consider the topographical properties of the noncommutativity space-space and phase-phase or noncommutative phase space (NCPS) have a clear effect on the various physical properties of relativistic and nonrelativistic quantum systems [21–31]. We have had a rich personal experience in studying many potentials related to the quark system in NCPS, such as the energy-dependent potential for heavy quarkouniom [32], new modified potential containing Cornell, Gaussian and inverse square terms for describing heavy-light mesons [33],

modified quark-antiquark interaction potential [34], extended modified Cornell potential [35], extended nonrelativistic quark model [36]. We have investigated a nonrelativistic bound state solution at finite temperature using the sum of modified Cornell plus inverse quadratic potential and modified Cornell potential [37, 38]. In 2021, we have obtained solutions of the Klein-Gordon equation for the modified central complex potential in the symmetries of noncommutative quantum mechanics for the bosonic particles like heavy-light mesons $Q\overline{q}\,(Q=c,\ q=u,d,c)$ and the quarkonium system $q\overline{q}\,(q=c,\ b,\ s)$ [39]. Motived of the previous works in ordinary QM and NCPS we hope to investigate the TRMP in 3D-NCPS to obtain new applications on the microscopic scale and from achieving more scientific knowledge of elementary particles in the field of nano-scales. The nonrelativistic energy levels under the improved trigonometric Rosen-Morse potential model (ITRMP model) have not been obtained yet in the 3D-NCPS symmetries, we propose a new version of the ITRMP model, as follows:

$$V_{nc}^{\text{trm}}(r)=V_{\text{trm}}(r)+\left(\frac{C}{r^4}-\frac{A}{2r^3}-\frac{B}{2r}-D\right)\mathbf{L\Theta}\,. \tag{2}$$

Here $\mathbf{L\Theta}$ is the coupling of the angular momentum operator $\mathbf{L}$ and NC properties $\mathbf{\Theta}$ which equal $L_x\Theta_{12}+L_y\Theta_{23}+L_z\Theta_{13}$, ($L_x$, L_y and L_z) are the three components while $\Theta_{\alpha\beta}=\dfrac{\theta_{\alpha\beta}}{2}$. $D=\dfrac{a\,(a+1)}{30\mu d^4}$, $B=\dfrac{b}{3\mu d^3}$, $A=\dfrac{b}{\mu d}$ and $C=\dfrac{a\,(a+1)}{2\mu}$ are the potential parameters, r is the distance between the two particles. The algebraic structure of nonrelativistic 3D-NCPS based to NC-canonical commutations relations in (Schrödinger, Heisenberg and interactions) pictures (SP, HP and IP), respectively, as follows (Throughout this paper, the natural units $c=\hbar=1$ will be used) (see, e.g., [40–48]):

$$\left[x_{\alpha S}^{nc}\,{}^{*}{}_{;}p_{\beta S}^{nc}\right]=\left[x_{\alpha I}^{nc}(t)\,{}^{*}{}_{;}p_{\beta I}^{nc}(t)\right]=\left[x_{\alpha H}^{nc}(t)\,{}^{*}{}_{;}p_{\beta H}^{nc}(t)\right]=i\hbar_{\text{eff}}\delta_{\alpha\beta} \tag{3}$$

$$\left[x_{\alpha S}^{nc}\,{}^{*}{}_{;}x_{\beta S}^{nc}\right]=\left[x_{\alpha I}^{nc}(t)\,{}^{*}{}_{;}x_{\beta I}^{nc}(t)\right]=\left[x_{\alpha H}^{nc}(t)\,{}^{*}{}_{;}x_{\beta H}^{nc}(t)\right]=i\theta_{\alpha\beta} \tag{4}$$

$$\left[p_{\alpha S}^{nc}\,{}^{*}{}_{;}p_{\beta S}^{nc}\right]=\left[p_{\alpha I}^{nc}(t)\,{}^{*}{}_{;}p_{\beta I}^{nc}(t)\right]=\left[p_{\alpha H}^{nc}(t)\,{}^{*}{}_{;}p_{\beta H}^{nc}(t)\right]=i\eta_{\alpha\beta} \tag{5}$$

The very small two parameters $\left(\theta^{\alpha\beta},\eta^{\alpha\beta}\right)=\epsilon^{\alpha\beta}(\theta,\eta)$ (compared to the energy) are elements of two antisymmetric real matrixes with dimensions of (length)^2 and (momentum)^2, respectively while $\epsilon^{\alpha\beta}$ presenting an antisymmetric number and $\hbar_{\text{eff}}=\hbar\,(1+\text{Tr}\,(\theta\eta/4))$ and $\hbar$ are the effective Planck constant and the Planck constant. Furthermore, the notation $*$ denotes the star products, which is generalized between two arbitrary functions $h\,(x,p)\,g\,(x,p)$ to the form $h\,(x,p)*g\,(x,p)\equiv(h*g)\,(x,p)$ in ordinary 3D-NCPS (see, e.g., [49–58]:

$$(h*g)(x,p)=\left(hg-\frac{\alpha}{2}\theta^{\alpha\beta}\partial_{x^\alpha}f\partial_{x^\beta}g-\frac{\alpha}{2}\eta^{\alpha\beta}\partial_{p^\alpha}h\partial_{p^\beta}g\right)(x,p)+O(\eta^2,\theta^2)\,. \tag{6}$$

Here $(\partial_{x^\alpha}, \partial_{p^\alpha}) = \left(\frac{\partial}{\partial_{x^\alpha}}, \frac{\partial}{\partial_{p^\alpha}}\right)$. This allows the formation of a scale of two space and phase cells with volumes $(\tau_{ns}^{nc3}, \tau_{p}^{nc3}) = (\theta^{3/2}, \eta^{3/2})$ respectively. On the other hand, Eq. (6) allows us to satisfy the postulated algebra in Eqs. (3), (4) and (5). The second and the third terms in the above equation are the effects of (space-space) and (phase-phase) noncommutativity properties, respectively. The aim of this paper to present approximate solutions of deformed Schrödinger equation DSE with the ITRMP model in 3D-NCPS symmetries, using the generalized Bopp's shift method (GBSM), in addition to the standard perturbation theory (SPT).

1.3 Outline

The organization scheme of the recent work is given as follows: In the next section, we briefly review the ordinary SE with TRMP model. We divided the third section into subsections, the first one reserved to the physical and mathematical model for ITRMP model in 3D-NCPS by applying GBSM, in the next subsection, we generate the spin-orbit Hamiltonian operator for the hydrogenic atoms and the HLM under the ITRMP model and by applying standard perturbation theory we find the corresponding corrected quantum spectrum of n^{th} excited levels in the framework of the global group 3D-NCPS and then, we derive the modified magnetic and rotational spectrums for the ITRMP model which produced with the effect of both perturbed Hamiltonians $H_z^{\text{trm}}\left(r, \lambda, \overline{\lambda}\right)$ and $H_{\text{pert}}^{\text{trm-rot}}$. In the fourth section, we resume the global spectrum and corresponding NC Hamiltonian operator for the ITRMP model and corresponding energy levels of the hydrogenic atoms (He^+, Li^{+2} and Be^+) and the heavy quarkonium system such as charmonium $c\overline{c}$ and bottomonium $b\overline{b}$ and we calculate the new mass spectra of ($c\overline{c}$, $b\overline{b}$) in 3D-NCPS symmetries. In the next section, we will conduct a comparative study between our results in 3D-NCPS and the obtained results in the commutative quantum mechanics for the hydrogenic atoms and the heavy quarkonium system. Finally, the achieved results are briefly summarized in the last section.

2 Overview of the Eigenfunctions and the Energy Eigenvalues for the TRMP Model in NRQM

In this section, we shall recall here the time-independent SE for the TRMP model $V_{\text{trm}}(r)$, which is an important short-range potential that behaves like a Coulomb potential for small values and decreases exponentially for large values presented in Eq. (1). For small values of r/d the TRMP takes the form [20]:

$$V_{\text{trm}}(r) = -\frac{A}{r} + Br + \frac{C}{r^2} + Dr^2\,. \tag{7}$$

If we consider a nonrelativistic virtual particle of reduced mass μ in a central potential like $V_{\text{trm}}(r)$, the quantum evolution of this particle governed with the

SE in the spherical coordinates (r,θ,φ) as follows:

$$i\hbar\frac{\partial\Psi\left(r,\theta,\varphi,t\right)}{\partial t}=\left(-\frac{\Delta}{2\mu}+V_{\text{trm}}\left(r\right)\right)\Psi\left(r,\theta,\varphi,t\right)$$
$$\Longrightarrow\frac{d^{2}R_{nl}\left(r\right)}{dr^{2}}+2\mu\left(E_{nl}^{\text{trm}}-V_{\text{eff}}^{\text{trm}}\left(r\right)\right)R_{nl}\left(r\right)=0 \quad (8)$$

Here $V_{\text{eff}}^{\text{trm}}\left(r\right)=-A/r+Br+C/r^{2}+Dr^{2}+l(l+1)/r^{2}$ is the effective potential, E_{nl}^{trm} is the eigenvalues of the TRMP model while $(n,\,l)$ are the radial and orbital angular momentum quantum numbers and Δ is the usual Laplacian operator. The authors of [20] used the analytical exact iteration method to find the complete wave function $\Psi\left(r,\theta,\varphi,t\right)=\dfrac{R_{nl}\left(r\right)}{r}Y_{l}^{m}\left(\theta,\varphi\right)e^{-\frac{\alpha}{\hbar}E_{nl}^{\text{trm}}t}$, as a function of exponential terms, $Y_{l}^{m}\left(\theta,\varphi\right)$ and the energy E_{nl}^{trm} of the potential (7) as follows:

for $n=0$:

$$\Psi_{0l}\left(r,\theta,\varphi,t\right)=N_{0l}r^{\frac{l'-1}{2}}\exp\Big(-\frac{\lambda_{0}}{2}r^{2}-\frac{\gamma_{0}}{2}r\Big)Y_{l}^{m}\left(\theta,\varphi\right)e^{-\frac{\alpha}{\hbar}E_{nl}^{\text{trm}}t} \quad (9)$$

for $n=1$:

$$\Psi_{1l}\left(r,\theta,\varphi\right)=N_{1l}(r-\alpha_{1}^{(1)})r^{\frac{l'-1}{2}}\exp\Big(-\frac{\lambda_{1}}{2}r^{2}-\frac{\gamma_{1}}{2}r\Big)$$
$$\times\,Y_{l}^{m}(\theta,\varphi)e^{-\frac{\alpha}{\hbar}E_{nl}^{\text{trm}}t} \quad (10)$$

for $n>1$:

$$\Psi_{nl}(r,\theta,\varphi)=N_{nl}\prod_{\alpha=1}^{n}(r-\alpha_{\alpha}^{(n)})r^{\frac{l'-1}{2}}\exp\Big(-\frac{\lambda_{1}}{2}r^{2}-\frac{\gamma_{1}}{2}r\Big)$$
$$\times\,Y_{l}^{m}(\theta,\varphi)e^{-\frac{\alpha}{\hbar}E_{nl}^{\text{trm}}t} \quad (11)$$

and

$$E_{nl}^{\text{trm}}=\frac{1}{2\mu}\left[\sqrt{\frac{a\left(a+1\right)}{15d^{4}}}\left(2+n+l'\right)-\frac{5b^{2}}{30a\left(a+1\right)d^{2}}\right]. \quad (12)$$

Here

$$l'=\sqrt{\left(l+\frac{1}{2}\right)^{2}+8\mu C}\,,$$
$$2\sqrt{-2\mu D}\alpha_{1}^{(1)}=2\mu A-2\frac{\mu B}{\sqrt{-2\mu D}}\left(\frac{1}{2}\left(1\pm l'\right)+1\right),$$
$$2\sqrt{-2\mu D}\sum_{\alpha=1}^{n}\alpha_{\alpha}^{(n)}=2\mu A-2\frac{\mu B}{\sqrt{-2\mu D}}\Big(\frac{1}{2}(1\pm l')+n\Big),$$

$$\lambda_0 = 2\sqrt{\frac{a(a+1)}{60d^4}}\,, \qquad \gamma_0 = 2\sqrt{\frac{5b^2}{30a(a+1)d^2}}\,,$$
$$\lambda_1 = \sqrt{\frac{a(a+1)}{15d^4}}\,, \qquad \gamma_1 = \frac{2b}{3d\sqrt{a(a+1)}}\,,$$

while N_{nl} is the normalization constant.

3 Solution of the DSE for Nonrelativistic ITRMP Model in 3D-NCPS Symmetries

3.1 Physical and mathematical model

In this subsection, we devote this part to studying the nonrelativistic ITRMP model $V_{nc}^{\text{trm}}(r)$, in 3D-NCPS symmetries. To perform this task the physical form DSE, it is necessary to replace ordinary three-dimensional Hamiltonian operators $H_{\text{trm}}(p,x)$, ordinary energy E_{nl}^{trm} and corresponding complex wave function $\Psi(\vec{r})$ in the symmetries of NRQM with new three-dimensional Hamiltonian operators $\widehat{H}_{nc}^{\text{trm}}(p_{nc},x_{nc})$, new unknown values E_{nc}^{trm} of energy and corresponding new complex wave function $\Psi(\vec{r}_{nc})$, respectively in 3D-NCPS symmetries. In addition, to replace the ordinary product by star product $*$, this allows us to construct the deformed Schrödinger DSE in 3D-NCPS symmetries as (see, e.g., [41–51]):

$$\widehat{H}_{nc}^{\text{trm}}(p_{nc},x_{nc})\Psi(\vec{r}_{nc}) = E_{nc}^{\text{trm}}\Psi(\vec{r_{nc}}) \Longrightarrow H_{nc}^{\text{trm}}(p,x)*\Psi(\vec{r}) = E_{nl}^{\text{trm}}\Psi(\vec{r}) \quad (13)$$

allows us to obtain the deformed RSE as follows:

$$\left[\frac{d^2}{dr^2} + 2\mu(E_{nl}^{\text{trm}} - V_{\text{eff}}^{\text{trm}}(r))\right] * R_{nl}(r) = 0\,. \quad (14)$$

The GBSM employed in the solutions enables us to explore an effective way of obtaining the ITRMP model in 3D-NCPS, it based on the following commutators (see, e.g., [58–62]):

$$\left[x_{\alpha S}^{nc} {}^{*}{;} p_{\beta S}^{nc}\right] = \left[x_{\alpha I}^{nc}(t) {}^{*}{;} p_{\beta I}^{nc}(t)\right] = \left[x_{\alpha H}^{nc}(t) {}^{*}{;} p_{\beta H}^{nc}(t)\right] = i\hbar_{\text{eff}}\delta_{\alpha\beta}\,, \quad (15)$$
$$\left[x_{\alpha S}^{nc} {}^{*}{;} x_{\beta S}^{nc}\right] = \left[x_{\alpha I}^{nc}(t) {}^{*}{;} x_{\beta I}^{nc}(t)\right] = \left[x_{\alpha H}^{nc}(t) {}^{*}{;} x_{\beta H}^{nc}(t)\right] = i\theta_{\alpha\beta}\,, \quad (16)$$
$$\left[p_{\alpha S}^{nc} {}^{*}{;} p_{\beta S}^{nc}\right] = \left[p_{\alpha I}^{nc}(t) {}^{*}{;} p_{\beta I}^{nc}(t)\right] = \left[p_{\alpha H}^{nc}(t) {}^{*}{;} p_{\beta H}^{nc}(t)\right] = i\eta_{\alpha\beta}\,. \quad (17)$$

The generalized positions and momentum coordinates $(x_{\alpha}^{nc}, p_{\alpha}^{nc})$ in 3D-NCPS depend on corresponding usual generalized positions and momentum coordinates (x_{α}, p_{α}) in NRQM by the following, respectively (see, e.g., [60–62]):

$$(x_{\alpha}, p_{\alpha}) \longrightarrow (x_{\alpha S}^{nc}, p_{\alpha S}^{nc}) = \left(x_{\alpha} - \sum_{\beta=1}^{3}\frac{\theta_{\alpha\beta}}{2}p_{\beta},\ p_{\alpha} + \sum_{\beta=1}^{3}\frac{\eta_{\alpha\beta}}{2}x_{\beta}\right) \quad (18)$$

The above equation allows us to obtain the two operators r_{nc}^2 and p_{nc}^2 in 3D-NCPS (see, e.g., [44,51,63,64]):

$$\left(r^2, p^2\right) \longrightarrow \left(r_{nc}^2, p_{nc}^2\right) = \left(r^2 - \mathbf{L\Theta}, p^2 + \mathbf{L}\eta\right) \tag{19}$$

The new coupling $\mathbf{L}\eta$ is $L_x\eta_{12} + L_y\eta_{23} + L_z\eta_{13}$ while $\mathbf{L\Theta}$ is defined in the first section. It should be noted that GBSM has been successfully applied at the relativistic level in the field of high energies using both the Klein-Gordon equation [65–78], Dirac equation [79–81] and Duffin-Kemmer-Petiau equation [82]. Thus, the reduced SE (without star product) can be written as

$$\left[\frac{d^2}{dr^2} + 2\mu\left(E_{nl}^{\text{trm}} - V_{\text{eff}}^{\text{trm}}\left(r_{nc}\right)\right)\right] R_{nl}\left(r\right) = 0\,. \tag{20}$$

The Hamiltonian operator $H_{nc}^{\text{trm}}\left(p, x\right)$ and the new effective potential $V_{\text{eff}-nc}^{\text{trm}}\left(r\right)$ for the ITRMP model can be expressed as

$$\begin{aligned} H_{nc}^{\text{trm}} &= H\left(x_\alpha \to x_\alpha^{nc},\, p_\alpha \to p_\alpha^{nc}\right), \\ V_{\text{eff}-nc}^{\text{trm}}\left(r\right) &= V_{nc}^{\text{trm}}\left(r_{nc}\right) + l(l+1)r_{nc}^{-2}\,. \end{aligned} \tag{21}$$

Now, we want to find the ITRMP model $V_{nc}^{\text{trm}}\left(r_{nc}\right)$ and $l(l+1)/r_{nc}^2$ in 3D-NCPS symmetries

$$\begin{aligned} V_{nc}^{\text{trm}}\left(r_{nc}\right) &= -\frac{A}{r_{nc}} + Br_{nc} + \frac{C}{r_{nc}^2} + Dr_{nc}^2\,, \\ l(l+1)r_{nc}^{-2} &= l(l+1)r^{-2} + l(l+1)r^{-4}\mathbf{L\Theta} + O\left(\Theta^2\right). \end{aligned} \tag{22}$$

After straightforward calculations, we can obtain the important terms ($-A/r_{nc}$, Br_{nc}, C/r_{nc}^2 and Dr_{nc}^2) which will be used to determine the ITRMP model in 3D-NCPS symmetries as

$$\begin{aligned} -\frac{A}{r_{nc}} &= -\frac{A}{r} - \frac{A\mathbf{L\Theta}}{2r^3} + O\left(\Theta^2\right), \quad Br_{nc} = Br - \frac{B\mathbf{L\Theta}}{2r} + O\left(\Theta^2\right), \\ \frac{C}{r_{nc}^2} &= \frac{C}{r^2} + \frac{C\mathbf{L\Theta}}{r^4} + O\left(\Theta^2\right), \qquad Dr_{nc}^2 = Dr^2 - D\mathbf{L\Theta} + O\left(\Theta^2\right). \end{aligned} \tag{23}$$

By making the substitution above equations (23) and (22) into Eq. (21), we find the global our working Hamiltonian operator $H_{nc}^{\text{trm}}\left(p, x\right)$ and the new effective potential $V_{\text{eff}-nc}^{\text{trm}}\left(r, \Theta\right)$ for the ITRMP model in 3D-NCPS symmetries

$$\begin{aligned} H_{nc}^{\text{trm}}\left(p, x\right) &= H_{\text{trm}}\left(p, x\right) + H_{\text{pert}}^{\text{trm}}\left(r, \Theta, \eta\right), \\ V_{\text{eff}-nc}^{\text{trm}}\left(r, \Theta\right) &= V_{\text{eff}}^{\text{trm}}\left(r\right) + V_{\text{pert}}^{\text{trm}}\left(r, \Theta\right), \end{aligned} \tag{24}$$

where the operator $H_{\text{trm}}\left(p, x\right)$ just the ordinary Hamiltonian operator in usual NRQM symmetries

$$H_{\text{trm}}\left(p, x\right) = \frac{p^2}{2\mu} - \frac{A}{r} + Br + \frac{C}{r^2} + Dr^2\,, \tag{25}$$

while $H^{\text{trm}}_{\text{pert}}(r,\Theta,\eta)$ and $V^{\text{trm}}_{\text{pert}}(r,\Theta)$ are proportional with two infinitesimal parameters (Θ and η as we shall see later in detail) and then we can be considered as perturbations terms in 3D-NCPS symmetries as

$$\begin{aligned} H^{\text{trm}}_{\text{pert}}(r,\Theta,\eta) &= \Big(\frac{C+l(l+1)}{r^4}-\frac{A}{2r^3}-\frac{B}{2r}-D\Big)\mathbf{L\Theta}+\frac{\mathbf{L}\eta}{2\mu}+O(\Theta^2,\eta^2), \\ V^{\text{trm}}_{\text{pert}}(r,\Theta) &= \Big(\frac{C+l(l+1)}{r^4}-\frac{A}{2r^3}-\frac{B}{2r}-D\Big)\mathbf{L\Theta}+O(\Theta^2). \end{aligned} \tag{26}$$

It is clear that the operator $H_{\text{trm}}(p,x)$ is just the Hamiltonian operator for hydrogenic atoms such as He^+, Li^{+2} and Be^+ in ordinary quantum mechanics if we consider $A=Ze^2$ and $-A/r$ is the main part while the generated part $H^{\text{trm}}_{\text{pert}}(r)$ appears as a result of the deformation of 3D-NCPS properties. Furthermore, this operator $H_{\text{trm}}(p,x)$ can be considered as a Hamiltonian for the heavy-light mesons in usual nonrelativistic quantum mechanics symmetries. In recent work, we can disregard the second term in the expression of $H^{\text{trm}}_{\text{pert}}(r,p)$ because we are interested in the corrections of first-order of the two parameters Θ and η.

3.2 Spin-orbit Hamiltonian operator for the hydrogenic atoms and the HLM under the ITRMP model

In this subsection, we want to derive the physical form of the induced Hamiltonian $H^{\text{trm}}_{\text{pert}}(r,p)$ due to space-phase noncommutativity. To achieve this goal, we replace $\mathbf{L\Theta}$ and $\mathbf{L}\eta$ by useful physical forms ($\epsilon\Theta\mathbf{LS}$ or $g_s\Theta\mathbf{LS}$) and ($\epsilon\eta\mathbf{LS}$ or $g_s\eta\mathbf{LS}$), respectively (see, e.g., [42, 51, 63, 64]):

$$H^{\text{trm}}_{\text{so}}(r,\Theta,\eta)=\left[\left(\frac{C+l(l+1)}{r^4}-\frac{A}{2r^3}-\frac{B}{2r}-D\right)\Theta+\frac{\eta}{2\mu}\right]\mathbf{LS}, \tag{27}$$

$$\begin{cases}\epsilon & \text{for hydrogenic atoms}\\ g_s & \text{for HLM systems}\end{cases}$$

Here $\Theta^2=\Theta^2_{12}+\Theta^2_{23}+\Theta^2_{13}$, $\eta^2=\eta^2_{12}+\eta^2_{23}+\eta^2_{13}$, $\mathbf{LS}\equiv\sum_{i=1}3L_{x_i}S_{x_i}$, $\epsilon\approx 1/137$ is the atomic fine structure constant and, g_s is the strong coupling constant and $\mathbf{S}$ denotes the spin of the hydrogenic atoms He^+, Li^{+2} and Be^+ or the heavy-light mesons. Thus, the spin-orbit interactions $H^{\text{trm}}_{\text{so}}(r,\Theta,\eta)$ appear automatically as a result of the deformation of the space phase. Now, physically, we can rewrite the quantum spin-orbit $\mathbf{LS}$ coupling as follows:

$$\mathbf{J}=\mathbf{L}+\mathbf{S}\Longrightarrow 2\mathbf{LS}=\mathbf{J}^2-\mathbf{L}^2-\mathbf{S}^2\,. \tag{28}$$

Here $\mathbf{J}$ is the total momentum of the hydrogenic atoms He^+, Li^{+2} and Be^+ or HLM systems. Substitution this equation into Eq. (27) yields

$$H^{\text{trm}}_{\text{so}}(r)=\left[\left(\frac{C+l(l+1)}{r^4}-\frac{A}{2r^3}-\frac{B}{2r}-D\right)\Theta+\frac{\eta}{2\mu}\right]G^2\,, \tag{29}$$

$$\begin{cases}\epsilon & \text{for hydrogenic atoms}\\ g_s & \text{for HLM systems}\end{cases},$$

where $G^2 = \mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2$. Our recent study can apply in two principal cases, the first case considers $A = Ze$, Z and e are the atomic numbers and the charge of the electron, the term $(-A/r)$ become an attractive Colombian potential, thus, we can consider the Hamiltonian described hydrogenic atoms He^+, Li^{+2} and Be^+ under the influence of external fields described by other terms $(Dr^2 + Br + C/r^2)$ in ordinary quantum mechanics and its extension 3D-NCPS. which allows us to the eigenvalues j of the total operator $\mathbf{j}$ can be obtained from the interval $|l - 1/2| \leq j \leq |l + 1/2|$. We have an occasion of determining two-sided bounds to the eigenvalues of the operator $\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2$ as follows:

$$k\left(j,l,s\right) = j(j+1) - l(l+1) - s(s+1) = \begin{cases} k_- \left(j = l - 1/2, l, s\right) \\ k_+ \left(j = l + 1/2, l, s\right) \end{cases} . \quad (30)$$

A second occasion of determining a diagonal matrix $H_{\text{so}}^{\text{trm}}$ of order (3×3) with diagonal elements $(H_{\text{so}}^{u-\text{trm}})_{11}$, $\left(H_{\text{so}}^{d-\text{trm}}\right)_{22}$ and $(H_{\text{so}}^{\text{trm}})_{33}$ as

$$\begin{aligned} (H_{\text{so}}^{u-\text{trm}})_{11} &= \epsilon k_+ \Big[\Big(\frac{C + l(l+1)}{r^4} - \frac{A}{2r^3} - \frac{B}{2r} - D\Big)\Theta + \frac{\eta}{2\mu}\Big] \\ &\quad \text{if } \; j = l + 1/2\,, \\ (H_{\text{so}}^{d-\text{trm}})_{22} &= \epsilon k_- \Big[\Big(\frac{C + l(l+1)}{r^4} - \frac{A}{2r^3} - \frac{B}{2r} - D\Big)\Theta + \frac{\eta}{2\mu}\Big] \\ &\quad \text{if } \; j = l - 1/2\,, \\ (H_{\text{so}}^{\text{trm}})_{33} &= 0\,. \end{aligned} \quad (31)$$

The non-null diagonal elements $(H_{\text{so}}^{u-\text{trm}})_{22}$ and $\left(H_{\text{so}}^{d-\text{trm}}\right)_{22}$ of the perturbed Hamiltonian operator $H_{\text{pert}}^{\text{trm}}$ can be influenced by the energy values E_{nl}^{trm} by creating two new additive corrections values of energies $\Delta E_{n-\text{so}}^{u-\text{trm}}$ and $\Delta E_{n-\text{so}}^{d-\text{trm}}$ as follows:

$$\begin{aligned} \Delta E_{n-\text{so}}^{u-\text{trm}} &= \langle\Psi|(H_{\text{so}}^{u-\text{trm}})_{11}|\Psi\rangle \\ &= \epsilon k_+ \langle\Psi|\Big[\Big(\frac{C + l(l+1)}{r^4} - \frac{A}{2r^3} - \frac{B}{2r} - D\Big)\Theta + \frac{\eta}{2\mu}\Big]|\Psi\rangle \\ \Delta E_{n-\text{so}}^{d-\text{trm}} &= \langle\Psi|(H_{\text{so}}^{d-\text{trm}})_{22}|\Psi\rangle \\ &= \epsilon k_- \langle\Psi|\Big[\Big(\frac{C + l(l+1)}{r^4} - \frac{A}{2r^3} - \frac{B}{2r} - D\Big)\Theta + \frac{\eta}{2\mu}\Big]|\Psi\rangle\,. \end{aligned} \quad (32)$$

The second case for the heavy-light mesons (HLM) for example scalar, vector, pseudoscalar, and pseudovector for (B, B_s, D and D_s) mesons, or the heavy quarkonium system such as charmonium $c\bar{c}$ and bottomonium $b\bar{b}$, that have the quark and antiquark flavor $q\bar{q}$, the eigenvalues of the spin-orbit coupling operator $\mathbf{LS}$ are $k\left(j,l,s\right) = j(j+1) - l(l+1) - s(s+1)$ corresponding $j = l + 1$ (spin upper), $j = l$ (spin middle) and $j = l - 1$ (spin lower), respectively. Then,

one can form a diagonal matrix for modified nonrelativistic quark-antiquark potential with diagonal elements $(H_{\rm so}^{\rm trm})_{11}$, $(H_{\rm so}^{\rm trm})_{22}$ and $(H_{\rm so}^{\rm trm})_{33}$ in 3D-NCPS symmetries

$$\begin{aligned}
(H_{\rm so}^{\rm trm})_{11} &= g_s k_1\Big[\Big(\frac{C+l(l+1)}{r^4}-\frac{A}{2r^3}-\frac{B}{2r}-D\Big)\Theta+\frac{\eta}{2\mu}\Big] \\
&\quad \text{for } j=l+1\,, \\
(H_{\rm so}^{\rm trm})_{22} &= g_s k_2\Big[\Big(\frac{C+l(l+1)}{r^4}-\frac{A}{2r^3}-\frac{B}{2r}-D\Big)\Theta+\frac{\eta}{2\mu}\Big] \\
&\quad \text{for } j=l\,, \\
(H_{\rm so}^{\rm trm})_{33} &= g_s k_3\Big[\Big(\frac{C+l(l+1)}{r^4}-\frac{A}{2r^3}-\frac{B}{2r}-D\Big)\Theta+\frac{\eta}{2\mu}\Big] \\
&\quad \text{for } j=l-1\,.
\end{aligned} \tag{33}$$

Here $2\,(k_1,k_2,k_3)\equiv(l,-2,-2l-2)$. The above non-null diagonal elements of the perturbed Hamiltonian operator $H_{\rm pert}^{\rm trm}$ can be influenced by the energy values $E_{nl}^{\rm trm}$ by creating three additive corrections values $\Delta E_{n-u}^{\rm trm}$, $\Delta E_{n-m}^{\rm trm}$ and $\Delta E_{n-l}^{\rm trm}$ as follows:

$$\begin{aligned}
\Delta E_{n-u}^{\rm trm} &= \langle\Psi|(H_{\rm so}^{\rm trm})_{11}|\Psi\rangle \\
&= g_s k_1\langle\Psi|\Big[\Big(\frac{C+l(l+1)}{r^4}-\frac{A}{2r^3}-\frac{B}{2r}-D\Big)\Theta+\frac{\eta}{2\mu}\Big]|\Psi\rangle, \\
\Delta E_{n-m}^{\rm trm} &= \langle\Psi|(H_{\rm so}^{\rm trm})_{22}|\Psi\rangle \\
&= g_s k_2\langle\Psi|\Big[\Big(\frac{C+l(l+1)}{r^4}-\frac{A}{2r^3}-\frac{B}{2r}-D\Big)\Theta+\frac{\eta}{2\mu}\Big]|\Psi\rangle, \\
\Delta E_{n-l}^{\rm trm} &= \langle\Psi|(H_{\rm so}^{\rm trm})_{33}|\Psi\rangle \\
&= g_s k_3\langle\Psi|\Big[\Big(\frac{C+l(l+1)}{r^4}-\frac{A}{2r^3}-\frac{B}{2r}-D\Big)\Theta+\frac{\eta}{2\mu}\Big]|\Psi\rangle.
\end{aligned} \tag{34}$$

We have seen previously that the induced spin-orbit $H_{\rm so}^{\rm trm}(r)$ is infinitesimal compared to the principal Hamiltonian operator $H_{\rm trm}(p,x)$ in NRQM for hydrogenic atoms (He^+, Li^{+2} and Be^+) and the heavy quarkonioum systems ($c\bar{c}$ and $b\bar{b}$) under the ITRMP model. This allows us to apply standard perturbation theory to determine the nonrelativistic energy corrections $E_{\rm so}^{\rm trm}$ of hydrogenic atoms (He^+, Li^{+2} and Be^+) and the ITRMP model at the first order of two infinitesimal parameters Θ and η due to noncommutativity space-space and phase-phase properties.

3.3 Bound state solution for the spin-orbit operator for hydrogenic atoms and the HLM under the ITRMP model in 3D-NCPS symmetries

The TRM potential model is extended by including new radial terms $(C+l(l+1))r^{-4}$, Ar^{-3} and Br^{-1} to become an ITRMP model in 3D-NCPS symmetries.

The additive part $H_{\text{pert}}^{\text{trm}}(r)$ (Eq. (24)) of the new Hamiltonian operator $H_{nc}^{\text{trm}}(r,p)$ is also proportional to the infinitesimal two parameters Θ and η. This allows us to consider the additive part $H_{\text{pert}}^{\text{trm}}$ as a perturbation potential compared with the main potential $H_{\text{trm}}(p,x)$ (Eq. (25)) in the symmetries of 3D-NCPS, that is, the inequality $H_{\text{pert}}^{\text{trm}} \ll H_{\text{trm}}$ has to become satisfied. That is all the physical justifications for applying the time-independent perturbation theory become satisfied. Allows us to give a complete prescription for determining the energy level of the generalized n^{th} excited states. Now, we apply the standard perturbative theory, in 3D-NCPS symmetries, we find the expectation values of $\langle 1/r^4\rangle_{(nlm)}^{\text{trm}}$, $\langle 1/r^3\rangle_{(nlm)}^{\text{trm}}$ and $\langle 1/r\rangle_{(nlm)}^{\text{trm}}$ taking into account the wave function which we have seen previously in Eq. (11). Thus, after straightforward calculations, we obtain the following results:

$$\left\langle \frac{1}{r^4} \right\rangle_{(nlm)}^{\text{trm}} = N_{nl}^2 \int_0^{+\infty} \Big(\prod_{\alpha=1}^{n} (r-\alpha_\alpha^{(n)}) \Big)^2 r^{(l'-1)-2} \times \exp(-\lambda_1 r^2 - \gamma_1 r) dr, \tag{35}$$

$$\left\langle \frac{1}{r^3} \right\rangle_{(nlm)}^{\text{trm}} = N_{nl}^2 \int_0^{+\infty} \Big(\prod_{\alpha=1}^{n} (r-\alpha_\alpha^{(n)}) \Big)^2 r^{(l'-1)-1} \times \exp\left(-\lambda_1 r^2 - \gamma_1 r\right) dr, \tag{36}$$

$$\left\langle \frac{1}{r} \right\rangle_{(nlm)}^{\text{trm}} = N_{nl}^2 \int_0^{+\infty} \Big(\prod_{\alpha=1}^{n} (r-\alpha_\alpha^{(n)}) \Big)^2 r^{(l'-1)+1} \times \exp\left(-\lambda_1 r^2 - \gamma_1 r\right) dr. \tag{37}$$

We have replaced $\iint Y_l^m(\theta,\varphi)\, Y_{l'}^{m'}(\theta,\varphi) \sin(\theta)\, d\theta d\varphi$ with its values $\delta_{ll'}\delta_{mm'}$ and in order to reduce the burden of writing, we used the abbreviation $\langle D\rangle_{(nlm)}$ denote to the $\langle n,l,m|\, D\, |n,l,m\rangle$ as short handnotation. For the ground state $n=0$, the above expectation values in Eqs. (35)–(37) reduce to the following simple form:

$$\left\langle \frac{1}{r^4} \right\rangle_{(nlm)}^{\text{trm}} = N_{0l}^2 \int_0^{+\infty} r^{(l'-2)-1} \exp(-\lambda_0 r^2 - \gamma_0 r) dr, \tag{38}$$

$$\left\langle \frac{1}{r^3} \right\rangle_{(nlm)}^{\text{trm}} = N_{0l}^2 \int_0^{+\infty} r^{(l'-1)-1} \exp(-\lambda_0 r^2 - \gamma_0 r) dr, \tag{39}$$

$$\left\langle \frac{1}{r} \right\rangle_{(nlm)}^{\text{trm}} = N_{0l}^2 \int_0^{+\infty} r^{(l'+1)-1} \exp(-\lambda_0 r^2 - \gamma_0 r) dr. \tag{40}$$

Comparing Eqs. (38)–(40) with the integral of the form [83]

$$\int_0^{+\infty} x^{\nu-1} \exp(-\lambda x^2 - \gamma x) dx = (2\lambda)^{-\frac{\nu}{2}} \Gamma(\nu) \exp\left(\frac{\gamma^2}{8\lambda}\right) D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right), \tag{41}$$

where $D_{-\nu}\left(\gamma/\sqrt{2\lambda}\right)$ and $\Gamma\left(\nu\right)$ denote to the Parabolic cylinder functions and Gamma function, respectively. After straightforward calculations, we can obtain the explicit results

$$\left\langle \frac{1}{r^4} \right\rangle^{\text{trm}}_{(0lm)} = N_{0l}^2 (2\lambda_0)^{-\frac{l'-2}{2}} \Gamma(l''-2) \exp\left(\frac{\gamma_0^2}{8\lambda_0}\right) D_{-(l'-2)}\left(\frac{\gamma_0}{\sqrt{2\lambda_0}}\right), \quad (42)$$

$$\left\langle \frac{1}{r^3} \right\rangle^{\text{trm}}_{(0lm)} = N_{0l}^2 (2\lambda_0)^{-\frac{l'-1}{2}} \Gamma(l''-1) \exp\left(\frac{\gamma_0^2}{8\lambda_0}\right) D_{-(l'-1)}\left(\frac{\gamma_0}{\sqrt{2\lambda_0}}\right), \quad (43)$$

$$\left\langle \frac{1}{r} \right\rangle^{\text{trm}}_{(0lm)} = N_{0l}^2 (2\lambda_0)^{-\frac{l'+1}{2}} \Gamma(l'+1) \exp\left(\frac{\gamma_0^2}{8\lambda_0}\right) D_{-(l'+1)}\left(\frac{\gamma_0}{\sqrt{2\lambda_0}}\right). \quad (44)$$

For the first excited state $n = 1$, the expectation values in Eqs. (35)–(37) are reduced to the following simple form:

$$\begin{aligned}
\left\langle \frac{1}{r^4} \right\rangle^{\text{trm}}_{(1lm)} &= N_{1l}^2 \int_0^{+\infty} (r^{l'-1} + \alpha_1^{(1)2} r^{(l'-2)-1} - 2\alpha_1^{(1)} r^{(l'-1)-1}) \\
&\quad \times \exp(-\lambda_1 r^2 - \gamma_1 r) dr, \\
\left\langle \frac{1}{r^3} \right\rangle^{\text{trm}}_{(1lm)} &= N_{1l}^2 \int_0^{+\infty} (r^{l'+1-1} + \alpha_1^{(1)2} r^{(l'-1)-1} - 2\alpha_1^{(1)} r^{l'-1}) \\
&\quad \times \exp(-\lambda_1 r^2 - \gamma_1 r) dr, \\
\left\langle \frac{1}{r} \right\rangle^{\text{trm}}_{(1lm)} &= N_{1l}^2 \int_0^{+\infty} (r^{l'+3-1} + \alpha_1^{(1)2} r^{(l'+1)+1} - 2\alpha_1^{(1)} r^{(l'+2)-1}) \\
&\quad \times \exp(-\lambda_1 r^2 - \gamma_1 r) dr.
\end{aligned} \quad (45)$$

Comparing Eqs. (45) with the integral Eq. (41), we obtain the following results:

$$\left\langle \frac{1}{r^4} \right\rangle^{\text{trm}}_{(1lm)} = N_{1l}^{\prime 2} \Bigg\{ (2\lambda_1)^{-\frac{l'}{2}} \Gamma\left(l'\right) D_{-l'}\left(\frac{\gamma_1}{\sqrt{2\lambda_1}}\right) + (l' \to l'-2) + (l' \to l'-1) \Bigg\}, \quad (46)$$

$$\left\langle \frac{1}{r^3} \right\rangle^{\text{trm}}_{(1lm)} = N_{1l}^{\prime 2} \Bigg\{ (2\lambda_1)^{-\frac{l'+1}{2}} \Gamma(l'+1) D_{-(l'+1)}\left(\frac{\gamma_1}{\sqrt{2\lambda_1}}\right) + (l''+1 \to l'-1) + (l'+1 \to l') \Bigg\}, \quad (47)$$

$$\Big\langle \frac{1}{r} \Big\rangle^{\text{trm}}_{(1lm)} = N'^{2}_{1l}\Bigg\{ (2\lambda_1)^{-\frac{l'+3}{2}}\Gamma(l'+3)D_{-(l'+3)}\Big(\frac{\gamma_1}{\sqrt{2\lambda_1}}\Big) + (l'+3\to l'+1)+(l'+3\to l'+2)\Bigg\} \quad (48)$$

with $N'^{2}_{1l} = N^{2}_{1l}\exp(\gamma_1^2/8\lambda_1)$. The principal goal of this subsection is to determine the energy spectrum $\Delta E^{u-\text{trm}}_{n-\text{so}}(k_+,n,\mu,b,d,j,l,s) \equiv \Delta E^{u-\text{trm}}_{n-\text{so}}$ and $\Delta E^{d-\text{trm}}_{n-\text{so}}(k_-,n,\mu,b,d,j,l,s) \equiv \Delta E^{d-\text{trm}}_{n-\text{so}}$ which produced to $H^{\text{trm}}_{\text{so}}$ corresponding to $j = l+1/2$ and $j = l-1/2$ at the first order of two parameters Θ and η for hydrogenic atoms He^+, Li^{+2} and Be^+ for $(n,l\,)$ states by applying standard perturbation theory and through the structure constants which specified the dimensionality of the ITRMP model. Thus, we obtain the following results for the ground state and first excited state, respectively:

$$\begin{pmatrix}\Delta E^{u-\text{trm}}_{0-\text{so}}\\ \Delta E^{d-\text{trm}}_{0-\text{so}}\end{pmatrix} = \epsilon\Big(\chi\,(0,\mu,b,d)\,\Theta+\frac{\eta}{2\mu}\Big)\begin{pmatrix}k_+ \text{ for } j=l+1/2\\ k_- \text{ for } j=l-1/2\end{pmatrix} \quad (49)$$

$$\begin{pmatrix}\Delta E^{u-\text{trm}}_{1-\text{so}}\\ \Delta E^{d-\text{trm}}_{1-\text{so}}\end{pmatrix} = \epsilon\Big(\chi\,(1,\mu,b,d)\,\Theta+\frac{\eta}{2\mu}\Big)\begin{pmatrix}k_+ \text{ for } j=l+1/2\\ k_- \text{ for } j=l-1/2\end{pmatrix} \quad (50)$$

with

$$\chi\,(n,\mu,b,d) = (C+l(l+1))\Big\langle\frac{1}{r^4}\Big\rangle^{\text{trm}}_{(nlm)} - \frac{A}{2}\Big\langle\frac{1}{r^3}\Big\rangle^{\text{trm}}_{(nlm)} - \frac{B}{2}\Big\langle\frac{1}{r}\Big\rangle_{(nlm)} - D\,. \quad (51)$$

This allows us to generalize the above results to the case of n^{th} excited states in 3D-NCPS symmetries for the hydrogenic atoms He^+, Li^{+2} and Be^+ as follows:

$$\begin{pmatrix}\Delta E^{u-\text{trm}}_{n-\text{so}}\\ \Delta E^{d-\text{trm}}_{n-\text{so}}\end{pmatrix} = \epsilon\Big(\chi(n,\mu,b,d)\Theta+\frac{\eta}{2\mu}\Big)\begin{pmatrix}k_+ \text{ for } j=l+1/2\\ k_- \text{ for } j=l-1/2\end{pmatrix}. \quad (52)$$

For the heavy quarkonioum systems ($c\overline{c}$ and $b\bar{b}$) which quarks and antiquarks of the same system ($Q\overline{Q}$), the eigenvalues of the spin-orbit coupling, we obtain the following results, for the ground state and first excited state, respectively:

$$\begin{aligned}\Delta E^{\text{trm}}_{0u} &= g_s k_1(l)\Big(\chi(0,\mu,b,d)\Theta+\frac{\eta}{2\mu}\Big) \quad \text{for } j=l+1,\\ \Delta E^{\text{trm}}_{0m} &= g_s k_2(l)\Big(\chi(0,\mu,b,d)\Theta+\frac{\eta}{2\mu}\Big) \quad \text{for } j=l,\\ \Delta E^{\text{trm}}_{0l} &= g_s k_3(l)\Big(\chi(0,\mu,b,d)\Theta+\frac{\eta}{2\mu}\Big) \quad \text{for } j=l-1\end{aligned} \quad (53)$$

and

$$\begin{aligned}
\Delta E_{1g}^{\text{trm}} &= g_s k_1(l)\Big(\chi(1,\mu,b,d)\Theta + \frac{\eta}{2\mu}\Big) \quad \text{for } j = l+1,\\
\Delta E_{1m}^{\text{trm}} &= g_s k_2(l)\Big(\chi(1,\mu,b,d)\Theta + \frac{\eta}{2\mu}\Big) \quad \text{for } j = l, \qquad (54)\\
\Delta E_{1l}^{\text{trm}} &= g_s k_3(l)\Big(\chi(1,\mu,b,d)\Theta + \frac{\eta}{2\mu}\Big) \quad \text{for } j = l-1,
\end{aligned}$$

which allows us to generalize the above results to the case of n^{th} excited states in 3D-NCPS symmetries for the heavy quarkonioum systems, charmonioum $c\overline{c}$ and bottomonioum $b\overline{b}$ as follows:

$$\begin{aligned}
\Delta E_{nu}^{\text{trm}} &= g_s k_1(l)\Big(\chi(n,\mu,b,d)\Theta + \frac{\eta}{2\mu}\Big) \quad \text{for } j = l+1,\\
\Delta E_{nm}^{\text{trm}} &= g_s k_2(l)\Big(\chi(n,\mu,b,d)\Theta + \frac{\eta}{2\mu}\Big) \quad \text{for } j = l, \qquad (55)\\
\Delta E_{nl}^{\text{trm}} &= g_s k_3(l)\Big(\chi(n,\mu,b,d)\Theta + \frac{\eta}{2\mu}\Big) \quad \text{for } j = l-1.
\end{aligned}$$

3.4 Bound state solution for modified Zeeman effect for ITRMP model

In this subsection, having obtained the energy spectrum ($\Delta E_{n-\text{so}}^{u-\text{trm}}$ and $\Delta E_{n-\text{so}}^{d-\text{trm}}$) which produced to $H_{\text{so}}^{\text{trm}}$ corresponding to $j = l+1/2$ and $j = l-1/2$ at the first order of two parameters Θ and η for hydrogenic atoms for (n,l) states and the degenerated energy ($\Delta E_{n-g}^{\text{trm}}$, $\Delta E_{n-m}^{\text{trm}}$, $\Delta E_{n-l}^{\text{trm}}$) of the heavy quarkonioum systems ($c\overline{c}$ and $b\overline{b}$). Now, it is possible to obtain the second self proper symmetry for the ITRMP model. This physical phenomenon is induced automatically from the influence of an external uniform magnetic field $\aleph$, if we make the following two simultaneous transformations to ensure that previous calculations are not reputed

$$(\Theta,\eta) \longrightarrow (\lambda,\overline{\lambda})\aleph\,. \qquad (56)$$

Here λ and $\overline{\lambda}$ are just two infinitesimal real proportional constants, and to simplify calculations without compromising physical content we choose the magnetic field parallel with (Oz) axis. Then we transform the spin-orbit coupling to the new physical phenomena as follows:

$$\left[\left(\frac{C+l(l+1)}{r^4} - \frac{A}{2r^3} - \frac{B}{2r} - D\right)\mathbf{\Theta} + \frac{\eta}{2\mu}\right]\mathbf{L} \Longrightarrow \digamma(\lambda,\overline{\lambda},r)\aleph L_z \qquad (57)$$

with $\digamma(\lambda,\overline{\lambda},r) = f(r)\lambda + \overline{\lambda}/2\mu$. Here

$$f(r) = \frac{C+l(l+1)}{r^4} - \frac{A}{2r^3} - \frac{B}{2r} - D.$$

This allowed us to derive the modified magnetic Hamiltonian operator $H_z^{\text{trm}}\left(r,\lambda,\overline{\lambda}\right)$ for previous hydrogenic atoms under the ITRMP model in 3D-NCPS symmetries as

$$H_z^{\text{trm}}(r,\lambda,\overline{\lambda})=\digamma(\lambda,\overline{\lambda},r)H_{\text{mod}}^z\begin{cases}\epsilon & \text{for hydrogenic atoms}\\ g_s & \text{for HLM systems}\end{cases} \tag{58}$$

Here $H_{\text{mod}}^z=\aleph\mathbf{B}-H_z$ denote to Zeeman effect in nonrelativistic NCQM, while $H_z=-\vec{\aleph}\vec{\mathbf{S}}$ is just the ordinary Zeeman effect. To obtain the exact NC magnetic modifications of energy for the ground state, the first excited state and n^{th} excited states of the hydrogenic atoms He^+, Li^{+2} and Be^+ and the heavy quarkonioum systems under the ITRMP model $(\Delta E_{0-\text{mag}}^{hy-\text{trm}},\Delta E_{1-\text{mag}}^{hy-\text{trm}},\Delta E_{n-\text{mag}}^{hy-\text{trm}})$ and $(\Delta E_{0-\text{mag}}^{hlm-\text{trm}},\Delta E_{1-\text{mag}}^{hlm-\text{trm}},\Delta E_{1-\text{mag}}^{hlm-\text{trm}})$, we just replace $(k_+\,(k_-)$ or $k\,(l))$ and (Θ,η) in the Eqs. (52) and Eqs. (53)–(55) by the following parameters m and $(\lambda,\overline{\lambda})\aleph$, respectively:

$$\begin{pmatrix}\Delta E_{0\text{mag}}^{hy-\text{trm}}\\ E_{0\text{mag}}^{hlm-\text{trm}}\end{pmatrix}=\aleph\Big(\chi(0,\mu,b,d)\lambda+\frac{\overline{\lambda}}{2\mu}\Big)m\begin{pmatrix}\epsilon\text{ for hydrogenic atoms}\\ g_s\text{ for HLM systems}\end{pmatrix}, \tag{59}$$

$$\begin{pmatrix}\Delta E_{1\text{mag}}^{hy-\text{trm}}\\ E_{1\text{mag}}^{hlm-\text{trm}}\end{pmatrix}=\aleph\Big(\chi(1,\mu,b,d)\lambda+\frac{\overline{\lambda}}{2\mu}\Big)m\begin{pmatrix}\epsilon\text{ for hydrogenic atoms}\\ g_s\text{ for HLM systems}\end{pmatrix}, \tag{60}$$

$$\begin{pmatrix}\Delta E_{n\text{mag}}^{hy-\text{trm}}\\ E_{n\text{mag}}^{hlm-\text{trm}}\end{pmatrix}=\aleph\Big(\chi(n,\mu,b,d)\lambda+\frac{\overline{\lambda}}{2\mu}\Big)m\begin{pmatrix}\epsilon\text{ for hydrogenic atoms}\\ g_s\text{ for HLM systems}\end{pmatrix}. \tag{61}$$

We have $(-l\leq m\leq+l)$, which allows us to fix $(2l+1)$ values for discrete numbers m. It should be noted that the results obtained in Eq. (52) can find it by direct calculation $(\Delta E_{n-\text{mag}}^{hy-\text{trm}},\Delta E_{n-\text{mag}}^{hlm-\text{trm}})=\langle\Psi|H_z^{\text{trm}}(r,\lambda,\overline{\lambda})|\Psi\rangle$ that takes the following explicit relation:

$$\begin{aligned}\Delta E_{n\text{mag}}^{hy-\text{trm}}&=\epsilon\aleph m\int_0^{+\infty}\Big(\prod_{\alpha=1}^{n}(r-\alpha_\alpha^{(n)})\Big)^2r^{(l'-1)}\\&\quad\times\exp(-\lambda_1r^2-\gamma_1r)\digamma(\lambda,\overline{\lambda},r)dr,\\ \Delta E_{n\text{mag}}^{hlm-\text{trm}}&=g_s\aleph m\int_0^{+\infty}\Big(\prod_{\alpha=1}^{n}(r-\alpha_\alpha^{(n)})\Big)^2r^{(l'-1)}\\&\quad\times\exp(-\lambda_1r^2-\gamma_1r)\digamma(\lambda,\overline{\lambda},r)dr.\end{aligned} \tag{62}$$

Now, for our purposes, we are interested in finding a new third automatically important symmetry for the ITRMP model at zero temperature in DSE symmetries. This physical phenomenon is induced automatically from the influence of a perturbed effective potential $H_{\text{pert}}^{\text{trm}}(r,p)$ which we have seen in Eq. (26). We

discover these important physical phenomena when our studied system consists of non-interacting is considered as Fermi gas, it is formed from all the particles in their gaseous state (He^{+}, Li^{+2} and Be^{+}) undergoing rotation with angular velocity Ω if we make the following two simultaneous transformations to ensure that the previous calculations are not repeated

$$\left\{\begin{array}{l}\Theta \to \tau\mathbf{\Omega} \\ \eta \to \overline{\tau}\mathbf{\Omega}\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\mathbf{L}\Theta \to \chi\mathbf{L\Omega} \\ \mathbf{L}\eta \to \overline{\tau}\mathbf{L\Omega}\end{array}\right\}. \tag{63}$$

Here τ and $\overline{\tau}$ are just infinitesimal real proportional constants. We can express the effective potential $H_{\text{pert}}^{\text{trm}-\text{rot}}(r,p)$ which induced the rotational movements of the hydrogenic atoms and the heavy-light mesons as follows:

$$H_{\text{pert}}^{\text{trm}-\text{rot}}(r,p) = \Lambda(\tau,\overline{\tau},r)\mathbf{L\Omega}\begin{cases}\epsilon & : \text{ Hydrogenic atoms} \\ g_s & : \text{ HLM systems}\end{cases} \tag{64}$$

with $\Lambda(\tau,\overline{\tau},r) = f(r)\tau + \overline{\tau}/2\mu$. To simplify the calculations without compromising physical content, we choose the rotational velocity $\mathbf{\Omega}$ parallel to the (Oz) axis. Then we transform the spin-orbit coupling to the new physical phenomena as follows:

$$\Lambda(\tau,\overline{\tau},r)\mathbf{L\Omega} = \Lambda(\tau,\overline{\tau},r)\Omega L_z\,. \tag{65}$$

All of this data allows for the discovery of the new energy shift $\Delta E_{\text{trm}}^{f-\text{rot}}(n,\mu,b,d,\tau,\overline{\tau},m) \equiv \Delta E_{\text{trm}}^{f-\text{rot}}$ due to the perturbed Fermi gas effect $V_{\text{pert}}^{\text{trm}-\text{rot}}(r)$ which is generated automatically by the influence of the ITRMP model for the n^{th} excited state in DSE symmetries as follows:

$$E_{\text{trm}}^{f-\text{rot}} = \Big(\chi(n,\mu,b,d)\tau + \frac{\overline{\tau}}{2\mu}\Big)\Omega m\begin{cases}\epsilon & : \text{ Hydrogenic atoms} \\ g_s & : \text{ HLM systems}\end{cases} \tag{66}$$

It is worth mentioning that the authors of Ref. [84] studied rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gas in a two and three-dimensional space at zero temperature, but in this study, the rotational term was added to the Hamiltonian operator, in contrast to our case, where this rotation term $\Lambda(\tau,\overline{\tau},r)\mathbf{L\Omega}$ automatically appears due to the large symmetries resulting from the deformation of the space-phase.

4 Results and Discussion

In the previous subsections, we have obtained the nonrelativistic solution of DSE for the ITRMP model which is described by the Hamiltonian operator as given in Eq. (24) by using the generalized Bopp's shift method and standard perturbation theory. The energy eigenvalues are calculated in the 3D-NCPS. The deformed eigenenergies for the ground state, the first excited state and n^{th} excited states of

the hydrogenic atoms He^{+}, Li^{+2} and Be^{+} with spin-1/2 under the ITRMP model $E^{(u,d)hy}_{nc0}(0,\mu,b,d,j,l,m,s) \equiv E^{(u,d)hy}_{nc0}$, $E^{(u,d)\text{trm}}_{nc1}(1,\mu,b,d,j,l,m,s) \equiv E^{(u,d)hy}_{nc1}$ and $E^{(u,d)hy}_{nc-n}(n,\mu,b,d,j,l,m,s) \equiv E^{(u,d)hy}_{ncn}$. Furthermore, $E^{hlm}_{nu}(n,\mu,b,d,j,l,m,s) \equiv E^{hlm}_{nu}$, $E^{hlm}_{nm}(n,\mu,b,d,j,l,m,s) \equiv E^{hlm}_{nm}$ and $E^{hlm}_{n-l}(n,\mu,b,d,j,l,m,s) \equiv E^{hlm}_{nl}$ of the heavy quarkonioum systems ($c\bar{c}$ and $b\bar{b}$) with spin $(0\vee 1)$ obtained in this paper based on our original results presented in Eqs. (49), (50), (52), (53), (54), (55), (59), (60), (61) and (66), in addition to the ordinary energy for TERM model which presented in Eq. (12) take the form:

For the hydrogenic atoms He^{+}, Li^{+2} and Be^{+}:

$$\begin{aligned}
E^{(u,d)hy}_{nc0} &= E_{0l} + \Big(\chi_0(\aleph\lambda+\Omega\tau) + \frac{\aleph\overline{\lambda}+\Omega\overline{\tau}}{2\mu}\Big)\epsilon m \\
&\quad + \epsilon N^2_{0l}\Big(\chi_0\Theta + \frac{\eta}{2\mu}\Big)\begin{cases} k_+ & \text{for } j=l+1/2 \\ k_- & \text{for } j=l-1/2 \end{cases} \qquad (67)\\
E^{(u,d)hy}_{nc1} &= E_{1l} + \Big(\chi_1(\aleph\lambda+\Omega\tau) + \frac{\aleph\overline{\lambda}+\Omega\overline{\tau}}{2\mu}\Big)\epsilon m \\
&\quad + \epsilon N^2_{1l}\Big(\chi_1\Theta + \frac{\eta}{2\mu}\Big)\begin{cases} k_+ & \text{for } j=l+1/2 \\ k_- & \text{for } j=l-1/2 \end{cases} \qquad (68)\\
E^{(u,d)hy}_{ncn} &= \frac{1}{2\mu}\Big(\sqrt{\frac{a(a+1)}{15d^4}}(2+n+l') - \frac{b^2}{6a(a+1)d^2}\Big) \\
&\quad + \Big(\chi_n(\aleph\lambda+\Omega\tau) + \frac{\aleph\overline{\lambda}+\Omega\overline{\tau}}{2\mu}\Big)\epsilon m \\
&\quad + \epsilon N^2_{nl}\Big(\chi_n\Theta + \frac{\eta}{2\mu}\Big)\begin{cases} k_+ & \text{for } j=l+1/2 \\ k_- & \text{for } j=l-1/2 \end{cases} \qquad (69)
\end{aligned}$$

For the heavy quarkonioum systems ($c\bar{c}$ and $b\bar{b}$):

$$\begin{aligned}
E^{hlm}_{nu} &= E^{\text{trm}}_{nl} + g_s\Big(\chi_n(\aleph\lambda+\Omega\tau) + \frac{\aleph\overline{\lambda}+\Omega\overline{\tau}}{2\mu}\Big)m \\
&\quad + g_s k_1(l)\Big(\chi_n\Theta + \frac{\eta}{2\mu}\Big) \quad \text{for } j=l+1\,, \\
E^{hlm}_{nm} &= E^{\text{trm}}_{nl} + g_s\Big(\chi_n(\aleph\lambda+\Omega\tau) + \frac{\aleph\overline{\lambda}+\Omega\overline{\tau}}{2\mu}\Big)m \\
&\quad + g_s k_2(l)\Big(\chi_n\Theta + \frac{\eta}{2\mu}\Big) \quad \text{for } j=l\,, \\
E^{hlm}_{nl} &= E^{\text{trm}}_{nl} + g_s\Big(\chi_n(\aleph\lambda+\Omega\tau) + \frac{\aleph\overline{\lambda}+\Omega\overline{\tau}}{2\mu}\Big)m \\
&\quad + g_s k_3(l)\Big(\chi_n\Theta + \frac{\eta}{2\mu}\Big) \quad \text{for } j=l-1\,,
\end{aligned} \qquad (70)$$

where E_{0l} and E_{1l} are the energy of the ground state and the first excited state in the symmetries of quantum mechanics under the TRM model

$$E_{0l} = \frac{1}{2\mu}\Big[\sqrt{\frac{a(a+1)}{15d^4}}(2+l') - \frac{5b^2}{30a(a+1)d^2}\Big] \tag{71}$$

$$E_{1l} = \frac{1}{2\mu}\Big[\sqrt{\frac{a(a+1)}{15d^4}}(3+l') - \frac{5b^2}{30a(a+1)d^2}\Big]. \tag{72}$$

Here $\chi\,(n,\mu,b,d) \equiv \chi_n$, thus, the total energy $E^{(u,d)hy}_{nc-n}$ and (E^{hlm}_{n-g}, E^{hlm}_{n-m}, E^{hlm}_{n-l}) for the hydrogenic atoms and the heavy quarkonioum systems, respectively, under the ITRMP model in 3D-NCPS symmetries, is the sum of the ordinary part of the energy E^{trm}_{nl} and the two corrections of energy that are produced automatically with the effect of perturbed spin-orbit and modified Zeeman effect. This is one of the main objectives of our research. Finally, we end this section by introducing the important result of this work as:

Case 1: For the hydrogenic atoms He^{+}, Li^{+2} and Be^{+}:

$$\Big[H_{\text{trm}} + \epsilon F(\lambda,\overline{\lambda},r)H^{z}_{\text{mod}} + \epsilon\Big(f(r)(\tau\mathbf{L\Omega} + \mathbf{\Theta LS}) + \frac{\overline{\eta}\mathbf{LS} + \overline{\tau}\mathbf{L\Omega}}{2\mu}\Big)\Big]\frac{R_{nl}(r)}{r}Y^{m}_{l}(\theta,\varphi) = \left\{\begin{array}{l} E^{u,hy}_{nc-n} \text{ for } j = l+1/2 \\ E^{d,hy}_{nc-n} \text{ for } j = l-1/2 \end{array}\right\}\frac{R_{nl}(r)}{r}Y^{m}_{l}(\theta,\varphi). \tag{73}$$

Case 2: For the heavy quarkonioum systems ($c\overline{c}$ and $b\overline{b}$):

$$\Big[H_{\text{trm}} + g_s F(\lambda,\overline{\lambda},r)H^{z}_{\text{mod}} + g_s\Big(f(r)(\tau\mathbf{L\Omega} + \mathbf{\Theta LS}) + \frac{\overline{\eta}\mathbf{LS} + \overline{\tau}\mathbf{L\Omega}}{2\mu}\Big)\Big]\frac{R_{nl}(r)}{r}Y^{m}_{l}(\theta,\varphi) = \left\{\begin{array}{l} E^{hlm}_{n-g} \text{ for } j = l+1 \\ E^{hlm}_{n-m} \text{ for } j = l \\ E^{hlm}_{n-l} \text{ for } j = l-1 \end{array}\right\}\frac{R_{nl}(r)}{r}Y^{m}_{l}(\theta,\varphi). \tag{74}$$

This is one of the main motivations for the topic of this work. It is clear that the obtained eigenvalues of energies are real, which allows us to consider the NC diagonal Hamiltonian H^{trm}_{nc} as a Hermitian operator. In addition and regarding the previously obtained results (25), (29), (58) and (66), the global Hamiltonian operator, at first order in and with the ITRMP model for hydrogenic atoms for

(n,l) states takes the form as

$$H_{\text{trm}}^{nc}=H_{\text{trm}}+\left[f(r)\lambda+\frac{\overline{\lambda}}{2\mu}\right]H_{\text{mod}}^{z}$$
$$+\left(f(r)(\tau\mathbf{L\Omega}+\mathbf{\Theta LS})+\frac{\overline{\eta}\mathbf{LS}+\overline{\tau}\mathbf{L\Omega}}{2\mu}\right)\begin{cases}\epsilon \text{ for hydrogenic atoms}\\ g_s \text{ for HLM systems}\end{cases} \tag{75}$$

This is the equation for hydrogenic atoms He^{+}, Li^{+2} and Be^{+} and the heavy quarkonioum systems ($c\overline{c}$ and $b\bar{b}$) under the influence of the ITRMP model interactions. It should be pointed out that this treatment considers only the first-order terms in either Θ or η. The first two parts of Eq. (75) presents the Hamiltonian operator in the ordinary quantum mechanics for the TRM model, the third part is the modified Zeeman Hamiltonian operators while the last part is the combined two effects correspond the spin-orbit and the rotational Fermi operator for the ITRMP model, which is induced automatically by the NC properties of space and phase. It is evident to consider the atomic quantum number m can take $(2l+1)$ values and we have also two values for $j=l+1/2$ and $j=l-1/2$ corresponding to up and down polarities for the hydrogenic atoms He^{+}, Li^{+2} and Be^{+}. For the heavy quarkonioum systems ($c\overline{c}$ and $b\bar{b}$), we have also three values for $j=l\pm1$ and $j=l$. Thus, every state in the NRQM symmetry of energy for the ITRMP model will be replaced by $2(2l+1)$ a sub-state in 3D-NCPS symmetries. Thus, the total complete degeneracy of obtained energy level of the ITRMP model is obtained as a sum of all allowed values l. Total degeneracy is thus,

$$\underbrace{\sum_{l=1}^{n-1}2(2l+1)=2n^{2}}_{\text{NRQM\_symmetries}}\rightarrow\underbrace{2\sum_{l=1}^{n-1}2(2l+1)=4n^{2}}_{\text{3D\_NCPS under ITRMP}}\quad\text{for hydrogenic atoms} \tag{76}$$

$$\underbrace{\sum_{l=1}^{n-1}2(2l+1)=2n^{2}}_{\text{NRQM\_symmetries}}\rightarrow\underbrace{3\sum_{l=1}^{n-1}2(2l+1)=6n^{2}}_{\text{3D\_NCPS under ITRMP}}\quad\text{for HLM systems} \tag{77}$$

Now we're going to recalculate the mass spectra of $Q\overline{Q}\,(Q=b,c)$ under the ITRMP model. It is well known that the spin of charmonium and bottomonium equal two values (0 or 1), because it consists of quark and anti-quark. For spin-1, we have three values of j ($j_1=l+1$, $j_2=l$, $j_3=l-1$), which allows us corresponding three values $(k_1,k_2,k_3)=\frac{1}{2}(l,-2,-2l-2)$ and thus, we obtain three values of energy:

$$E_{nu}^{hlm}=\frac{1}{2\mu}\left(\sqrt{\frac{a\,(a+1)}{15d^{4}}}\,(2+n+l')-\frac{b^{2}}{6a\,(a+1)\,d^{2}}\right)+g_s(\chi_n\,(\aleph\lambda+\Omega\tau)$$
$$+\frac{\aleph\overline{\lambda}+\Omega\overline{\tau}}{2\mu})m+g_s l\left(\chi_n\Theta+\frac{\eta}{2\mu}\right)\ \text{for}\ j=l+1, \tag{78}$$

$$E_{nm}^{hlm}=\frac{1}{2\mu}\left(\sqrt{\frac{a\,(a+1)}{15d^4}}\,(2+n+l')-\frac{b^2}{6a\,(a+1)\,d^2}\right)+g_s(\chi_n\,(\aleph\lambda+\Omega\tau)$$
$$+\frac{\aleph\overline{\lambda}+\Omega\overline{\tau}}{2\mu})m-g_s\left(\chi_n\Theta+\frac{\eta}{2\mu}\right)\ \text{for}\ j=l, \tag{79}$$

$$E_{nl}^{hlm}=\frac{1}{2\mu}\left(\sqrt{\frac{a\,(a+1)}{15d^4}}\,(2+n+l')-\frac{b^2}{6a\,(a+1)\,d^2}\right)+g_s(\chi_n\,(\aleph\lambda+\Omega\tau)$$
$$+\frac{\aleph\overline{\lambda}+\Omega\overline{\tau}}{2\mu})m-g_s\,(l+1)\left(\chi_n\Theta+\frac{\eta}{2\mu}\right)\ \text{for}\ j=l-1. \tag{80}$$

In the symmetries of ordinary quantum mechanics the mass spectra $Q\overline{Q}\,(Q=b,c)$ obtained by applying the following formula [8, 10, 20]:

$$M=2m_Q+E_{nl}^{\text{trm}}\,. \tag{81}$$

Here m_Q is bare quark masses. Thus, the new unpolarized mass $M_{nc}^{\text{trm}}\,(s=1)$ with spin-1 of $Q\overline{Q}\,(Q=b,c)$ charmonium and bottomonium, in 3D-NCPS symmetries, becomes as follows:

$$M_{nc}^{\text{trm}}\,(s=1)=2m_Q+\frac{1}{3}\left(E_{nu}^{hlm}+E_{nm}^{hlm}+E_{nl}^{hlm}\right). \tag{82}$$

We have replaced E_{nl}^{trm} with unpolarized statistical energy $\frac{1}{3}(E_{ng}^{hlm}+E_{nm}^{hlm}+E_{nl}^{hlm})$ which represents physically the non-polarized energy (energy independent of spin). After a simple calculation, we obtain the δM,

$$\delta M(s=1)=g_s\Big(\chi_n(\aleph\lambda+\Omega\tau)+\frac{\aleph\overline{\lambda}+\Omega\overline{\tau}}{2\mu}\Big)m-\frac{2}{3}g_s\Big(\chi_n\Theta+\frac{\eta}{2\mu}\Big), \tag{83}$$

with $\delta M\,(s=1)\equiv M_{nc}^{\text{trm}}\,(s=1)-M$, M is the mass spectra of the heavy quarkonium system $Q\overline{Q}\,(Q=b,c)$ [20]

$$M=2m+\frac{1}{2\mu}\Big(\sqrt{\frac{a(a+1)}{15d^4}}\,\Big(2+2n\sqrt{8\mu C+4(l+1/2)^2}\Big)$$
$$-\frac{5b^2}{3a(a+1)d^2}\Big). \tag{84}$$

This is the noncommutativity contribution for the mass spectra of $Q\overline{Q}\,(Q=b,c)$ charmonium and bottomonium under an ITRMP model. For spin-0, we have only one value of $j=l$, allows us the values $k=0$ and thus, we obtain the energy:

$$E_{nc}^{hlm}=\frac{1}{2\mu}\Big(\sqrt{\frac{a(a+1)}{15d^4}}(2+n+l')-\frac{b^2}{6a(a+1)d^2}\Big)$$
$$+g_s\Big(\chi_n(\aleph\lambda+\Omega\tau)+\frac{\aleph\overline{\lambda}+\Omega\overline{\tau}}{2\mu}\Big)m\,. \tag{85}$$

Thus, the modified mass $\delta M\left(s=0\right) \equiv M_{nc}^{\text{trm}}\left(s=0\right)-M$ with spin-0 of $Q\overline{Q}\left(Q=b,c\right)$ charmonium and bottomonium, becomes as follows:

$$\delta M\left(s=0\right)=g_{s}\left(\chi_{n}\left(\aleph\lambda+\Omega\tau\right)+\frac{\aleph\overline{\lambda}+\Omega\overline{\tau}}{2\mu}\right)m\,. \tag{86}$$

We now look at important special cases and relationships between our recent results and some other existing results in our previous work. When we set (A,B,C,D) equal $(c,b,0,a)$ the ITRMP model reduces to the modified quark-antiquark interaction potential [34], it is easy to show that equations (24), (53), (54), (55) and (62) are reduced to the modified interaction $\left(H_{qp}^{nc},V_{qp}^{nc}\left(r\right)\right)$ of a particle in the extended nonrelativistic quark-antiquark potential and corresponding NC spectrum $(E_{nc}^{uqp},E_{nc}^{mqp},E_{nc}^{lqp})$, respectively [34]:

$$\begin{array}{l} V_{qp}(r)=ar^{2}+br-\dfrac{c}{r} \\ H_{qp}=-\dfrac{\boldsymbol{\Delta}}{2\mu}+V_{qp}(r) \end{array}\Rightarrow\begin{array}{l} V_{qp}^{nc}(r)=V_{qp}(r)+\left(\dfrac{c}{2r^{3}}-\dfrac{b}{2r}-a\right)\mathbf{L\Theta} \\ H_{qp}^{nc}=H_{qp}+\left(\dfrac{c}{2r^{3}}-\dfrac{b}{2r}-a\right)\mathbf{L\Theta}+\dfrac{\mathbf{L}\eta}{2\mu} \end{array} \tag{87}$$

and

$$E_{nl}^{qp}\Rightarrow\begin{cases} E_{nc}^{uqp}=E_{nl}^{qp}+g_{s}\aleph\left(\Xi_{n}\lambda+\dfrac{\overline{\lambda}}{2\mu}\right)m+g_{s}l\left(\Xi_{n}\Theta+\dfrac{\eta}{2\mu}\right) & \text{for } j=l+1, \\ E_{nc}^{mqp}=E_{nl}^{qp}+g_{s}\aleph\left(\Xi_{n}\lambda+\dfrac{\overline{\lambda}}{2\mu}\right)m-g_{s}\left(\Xi_{n}\Theta+\dfrac{\eta}{2\mu}\right) & \text{for } j=l, \\ E_{nc}^{lqp}=E_{nl}^{qp}+g_{s}\aleph\left(\Xi_{n}\lambda+\dfrac{\overline{\lambda}}{2\mu}\right)m-g_{s}(l+1)\left(\Xi_{n}\Theta+\dfrac{\eta}{2\mu}\right) & \text{for } j=l-1, \end{cases} \tag{88}$$

where

$$E_{nl}^{qp}=\sqrt{\frac{a}{2\mu}}\left(2n+2l+3\right)+\frac{\left(2n-n-1\right)b^{2}}{4\left(l+1\right)a} \tag{89}$$

is the complete energy spectrum for the quark-antiquark system in usual nonrelativistic quantum mechanics [2] and

$$\Xi_{n}\equiv\Xi\left(n,\mu,a,b,c\right)=\frac{c}{2}\left\langle\frac{1}{r^{3}}\right\rangle_{(nlm)}^{\text{trm}}-\frac{b}{2}\left\langle\frac{1}{r}\right\rangle_{(nlm)}-a$$

with condition $(\tau,\overline{\tau})\equiv(0,0)$.

5 A Comparative Study

In this section, we will discuss a comparative study between the results we obtained in the framework of 3D-NCPS and compared to its counterpart known in the literature:

For the hydrogenic atoms He^{+}, Li^{+2} and Be^{+}: By noting the new energy expression we got in Eq. (69), we note that the result of our study includes Eq. (12) as the main part in addition to two additive parts depending on the polarity up for $j = l + 1/2$ and dawn for $j = l - 1/2$ and the various magnetic and rotational effects resulting from the modified Zeeman effect and the proper rotational of Fermi effect. Furthermore, the total complete degeneracy is changed to become $4n^2$ instated the values $2n^2$ in ordinary quantum mechanics symmetries.

For the heavy quarkonioum systems ($c\bar{c}$ and $b\bar{b}$): By noting the new energy expression we got in Eq. (70), we note that the result of our study includes Eq. (12) as the main part in addition to three additive parts depending on the polarities corresponding $j = l + 1$ (spin upper), $j = l$ (spin middle) and $j = l - 1$ (spin lower), respectively and the various magnetic rotational properties resulting from the modified Zeeman effect and the Fermi effect. Furthermore, the total complete degeneracy is changed to become $6n^2$ instated the values $2n^2$ in ordinary quantum mechanics symmetries. Concerning the mass spectra $Q\overline{Q}\,(Q = b, c)$ which studied in the framework of quantum mechanics known in the literature [20] its value will change within 3D-NCSP to be equal to its value expressed in Eq. (81) as a principal part plus the new uncertainty expressed in the two equations (83) and (86) according to the spin value.

Accordingly, the resulting topological properties which appear to the effect of the additional postulates affect the energy values of the studied physical systems such as the hydrogenic atoms He^{+}, Li^{+2} and Be^{+} and the heavy quarkonioum systems ($c\bar{c}$ and $b\bar{b}$), as their original values are considered within the framework of quantum mechanics known in the literature as main values plus corrections of small values compared to the original values.

It is possible to recover the results of commutative space in [20] under trigonometric Rosen–Morse potential when we consider two simultaneously are satisfied

$$\begin{matrix}(\Theta, \lambda, \tau) \to (0,0,0) \\ \left(\eta, \overline{\lambda}, \overline{\tau}\right) \to (0,0,0)\end{matrix}.$$

6 Conclusions

In conclusion, the nonrelativistic energy eigenvalues of the hydrogenic atoms He^{+}, $Li^{2+}$and Be^{+} and the heavy quarkonioum system calculations are carried out within the framework of the generalized Bopp's shift method using the standard perturbation theory in the framework of 3D-NCPS symmetries, we resume the main obtained results:

- The energy eigenvalues $E_{ncn}^{(u,d)hy}\,(n, \mu, a, b, c, j, l, m, s)$ of the bound states of the hydrogenic atoms He^{+}, $Li^{2+}$and Be^{+} under the ITRMP model with spin-1/2 for n^{th} excited states have been analytically found. The energy eigenvalues depend on $(\mu, a, b, c\,)$ parameters and the discrete atomic quantum numbers $(j, l, m, s\,)$.

- The energy eigenvalues $E_{nu}^{hlm}(n,\mu,a,b,c,j,l,m,s)$, $E_{nm}^{hlm}(n,\mu,a,b,c,\beta,l,m,s)$ and E_{nl}^{hlm} (n, μ, a, b, c, j, l, m, s) of the bound states of the heavy quarkonioum systems ($c\overline{c}$ and $b\overline{b}$) under the ITRMP model, with spin-s for excited states have been analytically found.
- The ordinary kinetic term $-\frac{\Delta}{2\mu}$ modified to the new form ($-\frac{\Delta}{2\mu}-\frac{\mathbf{L}\eta}{2\mu}-\frac{\mathbf{L}\overline{\tau}}{2\mu}-\frac{\mathbf{L}\overline{\lambda}}{2\mu}$) for the ITRMP model in 3D-NCPS symmetries.
- The Hamiltonian operator in 3D-NCPS symmetries $H_{\text{trm}}^{nc}\left(r,\Theta,\eta,\lambda,\overline{\lambda}\right)$ is the sum of the Hamiltonian operator of the ITRMP model $H_{\text{trm}}\left(p,x\right)$and three operators, the first one is the modified spin-orbit interaction $H_{\text{so}}^{\text{trm}}\left(r,\Theta,\eta\right)$, the second is the modified Zeeman operator $H_{z}^{\text{trm}}\left(r,\lambda,\overline{\lambda}\right)$ while the third operator $H_{\text{pert}}^{\text{trm}-\text{rot}}$ is the perturbed Fermi Hamiltonian for the heavy quarkonioum systems and the hydrogenic atoms.
- We have obtained the modified mass spectra of $Q\overline{Q}(Q=b,c)$ under the ITRMP model in the symmetries of 3D-NCPS symmetries which is the sum of usual mass under the TRMP model and a small term proportional with the noncommutativity parameters.
- We have replaced usual energy in NRQM E_{nl}^{trm} with unpolarized statistical energy $\frac{1}{3}\left(E_{nu}^{hlm}+E_{nm}^{hlm}+E_{nl}^{hlm}\right)$ which represents physically the non-polarized energy (energy independent of spin) to obtain the new unpolarized mass M_{nc}^{trm} with spin-(0 or 1) of $Q\overline{Q}$ $(Q=b,c)$ charmonium and bottomonium, in 3D-NCPS symmetries.

It has been shown that the DSE under the ITRMP model presents useful symmetry to standing the hydrogenic atoms He^{+}, Li^{2+} and Be^{+} and the heavy quarkonioum systems, charmonioum $c\overline{c}$ and bottomonioum $b\overline{b}$. It should be noted that the results obtained in this research would be identical with corresponding results in ordinary quantum mechanics when the two limits $(\Theta,\lambda,\tau)\rightarrow(0,0,0)$ and $\left(\eta,\overline{\lambda},\overline{\tau}\right)\rightarrow(0,0,0)$ are applied simultaneously.

Consequently, the study of the analytical solution of the three-dimensional SE for hydrogenic atoms and the heavy quarkonioum systems under the ITRMP model in 3D-NCPS symmetries could provide valuable information in many physical fields, and open a new big window for profound theoretical and experimental research.

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