

Deflection of Light due to Spheroidal Oblate Static Objects

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Abstract. Deflection of light due to massive objects was predicted by Einstein in his general theory of relativity. This deflection of light has been calculated by many researchers in the past, for spherically symmetric objects. But, in reality, most of these gravitating objects are not spherical; instead they are ellipsoidal (oblate) in shape. The objective of the present work is to study theoretically the effect of this ellipticity on the trajectory of a light ray. In the present work, we obtain a converging series expression for the deflection of a light ray due to an ellipsoidal gravitating object, characterised by an ellipticity parameter. As a boundary condition, by setting the ellipticity parameter to be equal to zero, we get back the same expression for deflection as due to Schwarzschild black hole. It is also found that the additional contribution in deflection angle due to this ellipticity though small, but could be typically higher than the similar contribution caused by the rotation of a celestial object. Therefore for a precise estimate of the deflection due to a celestial object, the calculations presented here would be useful.

KEY WORDS: Schwarzschild-like solution, oblate mass, gravitational deflection, null geodesics, Hamilton-Jacobi equation.

1 Introduction

According to Einstein's general theory of relativity (GTR), gravitation is the manifestation of space-time geometry due to mass and energy [1]. As a consequence of such geometry of space-time, the deflection of light due to a massive object was obtained by Einstein himself [2]. Einstein's prediction was successfully tested for the first time by Dyson et al. [3]. Schwarzschild obtained the exact solution of the Einstein's field equation for the curvature of space-time due to spherically symmetric distribution of mass [4]. Reissner [5] and Nordström [6] independently obtained such a solution for a charged spherical massive object. After 47 years Kerr [7] obtained the exact solution for the rotating spherical mass which is given in terms of Kerr line element and later, Newman et al. [8] obtained the corresponding solution for rotating-charged spherical mass

which is known as Kerr-Newman line element. After Einstein calculated the expression for deflection of light due to the gravitational field of static spherical mass, other researchers subsequently extended these calculations to find the expressions for deflections under different space-time geometries. Below, we go through some relevant work.

Hagihara [9] studied the trajectory of light in the Schwarzschild field and obtained the Hamilton-Jacobi equation for the said field. Einstein [10] investigated the lens like nature of the stars that bend the trajectory of the light. Later, Darwin [11] calculated the deflection of light in strong gravitational field regime and obtained the result in logarithmic form. Then, Refsdal and Bondi [12] examined in detail the gravitational deflection of light and based on these suggested a method to obtain the mass of the gravitating object. Also Liebes [13] analysed the gravitational lensing and studied the properties of image formation due to such lensing. Luminet [14] worked on the effect of strong gravitational field on the trajectories of light rays and the appearance of images due to such strong field. Image formation due to large deflection angle has been studied by Ohanian [15] and it was shown that under strong field, the deflection angles of light can be expressed in terms of elliptical integrals. Further, the effect of Schwarzschild black hole in strong field regime has been also analysed by Virbhadrha and Ellis [16, 17] using the null geodesic approach. The same authors also studied the image formations around the optic axis of lens geometry. They also studied the so-called weakly and strongly naked singularities and found that the second one does not form relativistic images because it does not possess photon spheres in the static case. In addition, the importance of relativistic image due to gravitational lensing along with primary and secondary images have been investigated by Virbhadrha [18]. Then, for different black hole geometries, Bozza [19] studied the effect of strong gravitational field and extended the expression under strong deflection limit, so as to allow the complete capture of the photons by the black hole. Systematic study on the deflection angle of the light due to static and spherically symmetric objects have been reported by Keeton and Petters [20]. The authors showed that under weak deflection limit, the expression for deflection of light can be expanded in the Taylor series in order to have higher-order terms beyond the standard expression. Also, Iyer and Petters [21] have worked on both the strong and weak field conditions for gravitational deflection of light. They have reformulated the smooth transition from the strong to weak deflection limits and then compared their values with numerically integrated ones. Bozza [22] reviewed the theoretical development in gravitational deflection due to static and spherically symmetric fields both under strong and weak conditions. The author further extended the studies to include the effect of rotation of the gravitating object, on the trajectory of light ray. More recently, Huang [23] has revisited the derivations for the deflection of light coming from celestial sources.

For a rotating spherical mass, Carter [24] studied the geodesic equations in Kerr

geometry. Later, Bray [25] obtained an expression for the trajectory of a light ray in the Kerr field. Again, Sari [26] worked on the deflection of light due to a slowly rotating spherical mass. More recently, Bozza [27] calculated the lensing formula for Kerr black hole when the trajectory of the light is very close to the equatorial plane. A more exact expression for bending angle in the equatorial plane has obtained by Iyer and Hanse [28] and showed that bending of light propagating in prograde direction is greater than that in the retrograde direction. Aazami et al. [29] further studied the trajectories of light rays in equatorial plane and in the field perpendicular to it for Kerr geometry. In addition, Kraniotis [30] obtained an equation of motion of light in Kerr gravitational field for the arbitrary inclination to the equatorial plane in presence of cosmological constant (Λ). Also, Chakraborty and Sen [31] worked on the deflection of light in Kerr geometry propagating slightly above the equatorial plane. On the other hand, the concept of cosmological constant or dark energy became more important after the discovery of the accelerating expansion of the universe [32, 33]. Various models have been proposed by many researchers to explain such mysterious behaviour of the universe such as Λ CDM model, dynamic scalar fields etc. (e.g. [34–36]). Gravitational deflection of light in such an expanding universe was studied by Younas et al. [37] where they considered strong gravitational lensing due to black hole surrounded by quintessence (known as Kiselev black hole), a dark energy candidate. Here they found that the trajectories of light also depend on the quintessence parameter (σ) in Kiselev black hole. And also Azreg-Aïnou et al. [38] extended the studies for charged Kiselev black hole.

Further, Hasse and Pelrick [39] studied the image formation in Kerr-Newman geometry and Kraniotis [40] obtained an analytic solution for the deflection angle. In another work, a series expression for the deflection angle due to the Kerr-Newman object was obtained by Chakraborty and Sen [41].

Besides the standard null geodesic approach, some researchers have calculated the gravitational deflection of light by considering the curved space-time structure as equivalent to some optical medium. The concept of such an approach was given by Balazs [42]. Using such a method, Sen [43] obtained a more exact expression for deflection angle without any approximation for Schwarzschild geometry. Furthermore, Roy and Sen [44] calculated the deflection angle in the Kerr field using the same optical medium method.

All the above literature is based on the assumption that the gravitating objects are spherically symmetric, whereas most of the celestial objects are ellipsoidal (oblate) which may be because of their rotation or other factors. Therefore, understanding the significance of the ellipsoidal shape of the objects on the trajectories of light rays is important for precise calculation. In the past, Newton's inverse square law of gravitation was considered a universal law of gravitation and was capable to explain most of the observed natural phenomena. Using such law of gravitation, gravitational potentials due to non-spherical masses have been obtained by some authors like- Gamaw and Cleveland [45], Chatterjee and Sen-

gupta [46], Gupta [47]. While Hofmeister and Criss [48], Hofmeister et al. [49] discussed the implications of Newtonian gravity in the system of oblate spheroid celestial objects. Although Newton's law of gravity holds true in most cases, it fails to answer many gravitational events such as the deflection of light, the perihelion of Mercury, gravitational red-shift, etc. whereas GTR have proper answer for them. However, GTR doesn't completely falsify Newton's law whereas it is a more general theory that was obtained by considering gravity as the geometry of space-time rather than inverse square law force. As a result, Newtonian gravity became a limiting case of GTR [50]. Since Einstein's GTR is the exact theory of gravity, the gravitational field for ellipsoidal objects can be understood only from the solution of the Einstein field equation obtained for the curvature of space-time due to ellipsoidal objects. Erez and Rosen [51] obtained such an exterior solution of the Einstein field equation for non-spherical deformed mass for the first time which is known as Erez-Rosen (E-R) metric. Then this solution was further investigated by some other authors Doroshkevich et al. [52], Winicour et al. [53], Young and Coulter [54], Quevedo and Parkes [55]. Hartle and Thorne [56] obtained the general solutions for slowly rotating axially symmetric object. Besides these, a good number of cases with space-time metrics for the mass with multiple moments has been discussed by some recent authors Frutos-Alfaro and Soffel [57], Frutos-Alfaro et al. [58], Boshkayev et al. [59]. Nikouravan [60] obtained the Schwarzschild-like line element for space-time geometries due to static spheroidal oblate mass by performing coordinate transformation in presence of oblate mass. Using this line element, Nikouravan and Rawal [61] studied the possible role of oblateness on the the gravitational deflection of light. On the other hand, Bini et al. [62] investigated the trajectories of light rays in Erez-Rosen space-time and obtained deflection angle of light in weak-field limit.

For the first time EHT (Event Horizon Telescope) collaboration captured image of a black hole located at the center of M87 galaxy in 2019 which further confirmed the validity of Einstein's general theory of relativity [63]. By observing such behaviour of light near the black hole one can study the space-time structure near that object and able to test other modified theories of gravity. Strong gravity region near such objects may be the key point to explore the quantum nature of space-time. Considering such notion Liu et al. [64] obtained the deflection angle of light with loop quantum gravity approach for a black hole and compared with EHT observation results. Again Azreg-Aïnou et al. [65] studied the orbitals near the black hole in Einstein-Aether theory.

In the present work, we considered Einstein's general theory of relativity and studied the gravitational deflection of light using Nikouravan's solution in a more general way and obtained a series expression for the angle of deflection of light due to an oblate gravitating object. The motivation of this work is to understand the space-time structure near an oblate object and hence the trajectories of light rays. Here we obtained the deflection angle for light rays contained in the plane

of ellipticity (plane perpendicular to the minor axis of an oblate) due to static gravitating objects. Further, we compared the contributions of ellipticity in deflection angle with that which may be caused due to the typical rotation of a celestial objects.

2 Static Gravitational Field due to a Spheroidal Oblate Mass

As stated above our work is limited to the static (*i.e.* non-rotating) ellipsoidal (oblate) gravitating object. Let the center of mass of this object be the origin of the Cartesian coordinate system in space. We assume, the ellipsoid has axes with lengths α , β and γ . If $\alpha = \beta > \gamma$, the object is known as a spheroidal oblate. The gravitational field due to such an oblate mass can be studied in an oblate coordinate system [60]. Let us take the z -axis of our co-ordinate frame be oriented along γ and x and y -axes be along α and β axes respectively. Thus one can write : $x = \sqrt{r^2 + a^2} \sin \theta \cos \phi$, $y = \sqrt{r^2 + a^2} \sin \theta \sin \phi$, $z = r \cos \theta$, where r is the radial coordinate, θ and ϕ are the angular coordinates measured from z -axis and y -axis respectively. The parameter $a = \sqrt{\alpha^2 - \gamma^2}$ is known as the linear eccentricity of the ellipsoid.

The line element obtained by Nikouravan [60] for the space-time geometry due to the spheroidal oblate object is:

$$ds^2 = \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \frac{1}{\left(1 - \frac{2m}{r}\right)} \left[\frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} \right] dr^2 - (r^2 + a^2 \cos^2 \theta) d\theta^2 - (r^2 + a^2) \sin^2 \theta d\phi^2, \quad (1)$$

where $2m = 2MG/c^2 = r_g$ is the Schwarzschild radius; m is known as Gravitational radius and M is the actual mass of the object in physical unit. From this line element (1), we have the metric elements as

$$g_{tt} = 1 - \frac{2m}{r}, \quad (2)$$

$$g_{rr} = -\frac{1}{1 - \frac{2m}{r}} \left[\frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} \right], \quad (3)$$

$$g_{\theta\theta} = -(r^2 + a^2 \cos^2 \theta), \quad (4)$$

$$g_{\phi\phi} = -(r^2 + a^2) \sin^2 \theta. \quad (5)$$

In matrix form this can be written as

$$g_{ij} = \begin{pmatrix} 1 - \frac{2m}{r} & 0 & 0 & 0 \\ 0 & -\frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} & 0 & 0 \\ 0 & 0 & 1 - \frac{2m}{r} & 0 \\ 0 & 0 & 0 & -(r^2 + a^2) \sin^2 \theta \end{pmatrix} \quad (6)$$

Therefore, we have the contravariant form of the metric as (using $g^{ij} = \frac{\text{co-factor } g_{ij}}{|g_{ij}|}$)

$$g^{ij} = \begin{pmatrix} \frac{1}{1 - \frac{2m}{r}} & 0 & 0 & 0 \\ 0 & -\frac{(1 - \frac{2m}{r})(r^2 + a^2)}{r^2 + a^2 \cos^2 \theta} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2 + a^2 \cos^2 \theta} & 0 \\ 0 & 0 & 0 & -\frac{1}{(r^2 + a^2) \sin^2 \theta} \end{pmatrix} \quad (7)$$

2.1 General Hamilton-Jacobi equation

For the study of trajectory of light, we apply the null geodesic approach. Before going to the main formulation, let us work-out some important tasks. First of all, in the general theory of relativity, Lagrangian (per unit mass) of a system is given by Chandrasekhar [66].

$$\begin{aligned} f &= \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j \\ &= \frac{1}{2} [g_{tt} c^2 \dot{t}^2 - g_{rr} \dot{r}^2 - g_{\theta\theta} \dot{\theta}^2 - g_{\phi\phi} \dot{\phi}^2], \end{aligned} \quad (8)$$

where dot over t , r , θ and ϕ denotes the differentiation with respect to an affine parameter λ along the geodesic. (Affine parameter is defined along the path of the geodesic [1]. For time like geodesics, it is related to proper time and for space like geodesics, it is related to proper distance.)

Now, from the Euler-Lagrange equation, we have

$$\frac{d}{d\lambda} \left(\frac{\partial f}{\partial \dot{x}^i} \right) - \frac{\partial f}{\partial x^i} = 0.$$

As the Lagrangian is independent of t and ϕ coordinates, we obtain below the following equations:

$$\begin{aligned} \frac{d}{d\lambda} \left(\frac{\partial f}{\partial \dot{t}} \right) - \frac{\partial f}{\partial t} &= 0 \Rightarrow \frac{d}{d\lambda} \left(\frac{\partial f}{\partial \dot{t}} \right) - 0 = 0 \\ \Rightarrow \left(\frac{\partial f}{\partial \dot{t}} \right) &= \left(1 - \frac{2m}{r} \right) c \dot{t} = E \text{ (a constant)} \end{aligned} \quad (9)$$

$$\begin{aligned} \text{and } \frac{d}{d\lambda} \left(\frac{\partial f}{\partial \dot{\phi}} \right) - \frac{\partial f}{\partial \phi} &= 0 \Rightarrow \frac{d}{d\lambda} \left(\frac{\partial f}{\partial \dot{\phi}} \right) - 0 = 0 \\ \Rightarrow \left(\frac{\partial f}{\partial \dot{\phi}} \right) &= (r^2 + a^2) \sin^2 \theta \dot{\phi} = L \text{ (a constant)}. \end{aligned} \quad (10)$$

Deflection of Light due to Spheroidal Oblate Static Objects

Since E and L are the constants of motion, hence they are the conserved quantities in the respective coordinate system. These two constants E and L respectively represent the energy and the angular momentum (per unit mass) around the short γ axis.

And the corresponding canonical momenta for other two coordinates can be obtained from the relation $p_i = \frac{\partial f}{\partial \dot{x}^i} = g_{ij} \dot{x}^j$ as below:

$$p_r = \frac{1}{\left(1 - \frac{2m}{r}\right)} \left[\frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} \right] \dot{r} \quad (11)$$

$$\text{and } p_\theta = (r^2 + a^2 \cos^2 \theta) \dot{\theta} \quad (12)$$

Again, the Hamiltonian of the system is given by [66],

$$H(x^i, p_j) = p_i \dot{x}^i(p_j) - f(x^i, \dot{x}^i) = \frac{1}{2} g^{ij} p_i p_j \quad (13)$$

and the Hamilton-Jacobi equation with solution $S \equiv S(x^i, \lambda)$ is

$$H\left(x^i, \frac{\partial S}{\partial x^i}\right) + \frac{\partial S}{\partial \lambda} = 0. \quad (14)$$

As the Hamiltonian is a function of coordinate and momentum, and $\dot{x}^i = g^{ij} p_j$, so here we have:

$$\frac{\partial S}{\partial x^i} = p_i \quad (15)$$

$$\text{and } \dot{x}^i = \frac{dx^i}{d\lambda} = g^{ij} \frac{\partial S}{\partial x^i}. \quad (16)$$

Thus in the above, we have seen that the Lagrangian of the ellipsoidal field is independent of t and ϕ , and so the momentum associated with these coordinates are constants of motion. The null geodesic action with such constants of motion can be written as

$$S = -Et + L\phi + S_{r,\theta}(r, \theta), \quad (17)$$

where $S_{r,\theta}(r, \theta)$ is the action associated with only r and θ coordinates. The Hamilton-Jacobi equation with such action for ellipsoidal field can be obtained as follows:

$$\begin{aligned} & \frac{1}{2} g^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} + \frac{\partial S}{\partial \lambda} = 0 \\ \text{or } & -\frac{E^2(r^2 + a^2 \cos^2 \theta)}{c^2 \left(1 - \frac{2m}{r}\right)} + \left[\left(1 - \frac{2m}{r}\right) (r^2 + a^2) \right] \left(\frac{\partial S_{r,\theta}}{\partial r} \right)^2 + \left(\frac{\partial S_{r,\theta}}{\partial \theta} \right)^2 \\ & \quad + \frac{(r^2 + a^2 \cos^2 \theta) L^2}{(r^2 + a^2) \sin^2 \theta} = 0 \\ \text{or } & \left[\left(1 - \frac{2m}{r}\right) (r^2 + a^2) \right] \left(\frac{\partial S_{r,\theta}}{\partial r} \right)^2 + \left(\frac{\partial S_{r,\theta}}{\partial \theta} \right)^2 - F(r, \theta) = 0, \quad (18) \end{aligned}$$

where

$$F(r, \theta) = (r^2 + a^2 \cos^2 \theta) \left[\frac{E^2}{c^2 \left(1 - \frac{2m}{r}\right)} - \frac{L^2}{(r^2 + a^2) \sin^2 \theta} \right]. \quad (19)$$

Equation (18) is the general Hamilton-Jacobi equation corresponding to a light ray moving in the gravitational field of an oblate object.

3 Trajectory of Light in the Plane of Oblateness

Let the entire trajectory of the light is contained in the plane of oblateness, i.e. the plane with $\theta = \pi/2$ (which is also the x - y plane). Thus, from Eqs. (16) and (19) we have,

$$F(r, \pi/2) = r^2 \left[\frac{E^2}{c^2(1 - 2m/r)} - \frac{L^2}{(r^2 + a^2)} \right] \quad (20)$$

$$L = (r^2 + a^2) \dot{\phi} \quad (21)$$

$$\text{and } \left(\frac{\partial S_{r,\theta}}{\partial \theta} \right)^2 = 0 \quad (22)$$

Therefore, the Hamilton-Jacobi equation (18) will be as follows:

$$(1 - 2m/r)(r^2 + a^2) \left(\frac{\partial S_{r,\theta}}{\partial r} \right)^2 = r^2 \left[\frac{E^2}{c^2(1 - 2m/r)} - \frac{L^2}{(r^2 + a^2)} \right]$$

$$\text{or } \left(\frac{\partial S_{r,\theta}}{\partial r} \right)^2 = r^2 \frac{\frac{E^2}{c^2(1 - 2m/r)} - \frac{L^2}{(r^2 + a^2)}}{(1 - 2m/r)(r^2 + a^2)} \quad (23)$$

And hence we can obtain radial geodesic from Eq. (16) as:

$$\dot{r}^2 = \left(\frac{dr}{d\lambda} \right)^2 = \left(g^{rr} \frac{\partial S_{r,\theta}}{\partial r} \right)^2,$$

$$\dot{r}^2 = (1 - 2m/r)^2 \left[\frac{r^2 + a^2}{r^2} \right]^2 r^2 \frac{\frac{E^2}{c^2(1 - 2m/r)} - \frac{L^2}{(r^2 + a^2)}}{(1 - 2m/r)(r^2 + a^2)}$$

$$= L^2(1 - 2m/r) \left[\frac{r^2 + a^2}{r^2} \right] \left[\frac{E^2/c^2 L^2}{(1 - 2m/r)} - \frac{1}{(r^2 + a^2)} \right], \quad (24)$$

$$\text{or } \dot{r} = \pm L \sqrt{\frac{\frac{1}{b^2} - \frac{(1 - 2m/r)}{(r^2 + a^2)}}{(1 - 2m/r)(1 - 2m/r)^{-1} \left[\frac{r^2 + a^2}{r^2} \right]^{-1}}}$$

$$= \pm L \sqrt{\left(\frac{1}{b^2} - \frac{1 - 2m/r}{r^2 + a^2} \right) / \left(\frac{r^2}{r^2 + a^2} \right)}, \quad (25)$$

Deflection of Light due to Spheroidal Oblate Static Objects

where the term denoted by $b = cL/E$ in the above equation is known as the apparent impact parameter or simply impact parameter [67], in which L and E are angular momentum and energy of the photon respectively, as stated earlier. Since the light is propagating with momentum p and energy E having speed c , we can write $E = pc$ and also $L = pb \Rightarrow b = cL/E$. This impact parameter (b) can be identified as the perpendicular distance between the centre of the gravitating object and the trajectory of the light ray. If the light is travelling parallel to the x -direction and contained in the $x - y$ plane, then this impact parameter is simply represented by the y - coordinate of light ray had there been no gravitational field.

3.1 Closest approach

The closest approach of the light ray (r_0) is the shortest distance between the center of gravitating object and the trajectory of the light. It is different from the impact parameter (b), as the impact parameter is the measure of the distance between center of the object and the trajectory of light when there is no gravitational field. The relation between b and r_0 can be obtained by taking the condition for the extremum value of Eq. (25). That is, if the light ray approaches to the closest distance $r = r_0$ (at affine parameter $\lambda = \lambda_0$); then $\dot{r} = 0$ and from the above Eq. (25) we can write

$$\frac{1}{b^2} = \frac{1 - 2m/r_0}{r_0^2 + a^2}, \quad (26)$$

$$\text{or } r_0^3 - r_0(b^2 - a^2) + 2mb^2 = 0. \quad (27)$$

The above Eq. (27) is a cubic equation in r_0 . The solution of this cubic equation can be obtained by using trigonometric identity:

$$4 \cos^3 \vartheta - 3 \cos \vartheta - \cos(3\vartheta) = 0. \quad (28)$$

For that, we considered $r_0 = n \cos \vartheta$; where n is a constant. Substituting this into Eq. (27), we obtain

$$r_0^3 - (b^2 - a^2)r_0 + 2mb^2 = n^3 \cos^3 \vartheta - (b^2 - a^2)n \cos \vartheta + 2mb^2 = 0. \quad (29)$$

Now, comparing Eqs. (28) and (29), we get

$$\frac{4}{n^3} = \frac{3}{(b^2 - a^2)n} = -\frac{\cos(3\vartheta)}{2mb^2}.$$

After simplification, we can write

$$n = 2\sqrt{\frac{(b^2 - a^2)}{3}}, \quad \text{and} \quad \cos(3\vartheta) = -\frac{3(2mb^2)}{(b^2 - a^2)n}$$

$$\Rightarrow \vartheta = \frac{1}{3} \cos^{-1} \left(\frac{3(2mb^2)}{-(b^2 - a^2)n} \right) = \frac{1}{3} \cos^{-1} \left(\frac{3(2mb^2)}{-2(b^2 - a^2)} \sqrt{\frac{3}{(b^2 - a^2)}} \right).$$

Therefore, the root of the Eq. (27) is

$$r_0 = n \cos \vartheta = 2\sqrt{\frac{(b^2 - a^2)}{3}} \cos \left[\frac{1}{3} \cos^{-1} \left(\frac{3(2mb^2)}{-2(b^2 - a^2)} \sqrt{\frac{3}{(b^2 - a^2)}} \right) \right]. \quad (30)$$

Since the value of $\cos(3\vartheta)$ remains unchanged by substituting $\vartheta_1 = \vartheta + 2\pi/3$ and $\vartheta_2 = \vartheta + 4\pi/3$ in the place of ϑ , we get that the cubic equation (27) has three roots given by $n \cos \vartheta$, $n \cos \vartheta_1$ and $n \cos \vartheta_2$. These roots can be expressed in a single general form by considering k as the number having values 0, 1, 2 as below:

$$r_0 = 2\sqrt{\frac{b^2 - a^2}{3}} \cos \left[\frac{1}{3} \cos^{-1} \left(\frac{3(2mb^2)}{-2(b^2 - a^2)} \sqrt{\frac{3}{b^2 - a^2}} \right) + \frac{2\pi k}{3} \right]. \quad (31)$$

We note that, the light is not captured while approaching the gravitating object and instead it has a point of bending at the largest value of r_0 . This indicates the closest approach corresponds to r_0 as the largest root [67] of Eq. (27). The largest value of r_0 can be obtained by putting $k = 0$ in Eq. (31). By plotting the above Eq. (31) for r_0 with respect to k from 0 \rightarrow 2 we can find its largest value at $k = 0$, as shown in Figure 1.

Hence, we obtain the expression for the closest approach of light as below:

$$\frac{r_0}{b_0} = \frac{2}{\sqrt{3}} \cos \left[\frac{1}{3} \cos^{-1} \left(-\frac{3^{3/2}m}{b^*} \right) \right]. \quad (32)$$

Here, we put $\sqrt{b^2 - a^2} = b_0$ and $b(1 - a^2/b^2)^{3/2} = b^*$. This Eq. (32) is the general expression for the closest approach in terms of impact parameter (b).

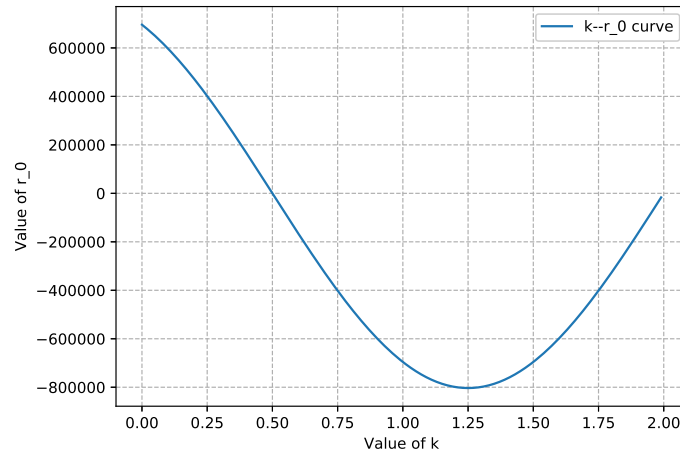


Figure 1. Characteristic curve of r_0 i.e. (k - r_0 curve).

Deflection of Light due to Spheroidal Oblate Static Objects

For weak field condition, we should have $m/b^* \ll 1$. Hence we can expand the right hand side of Eq. (32) in Taylor series in the power of $\frac{m}{b(1-a^2/b^2)^{3/2}} = \frac{m}{b^*}$ about the point at 0 as below:

$$\begin{aligned} \frac{r_0}{b_0} &= 1 - \frac{m}{b^*} - \frac{3}{2} \left(\frac{m}{b^*}\right) - 4 \left(\frac{m}{b^*}\right) + \dots \\ \text{or } r_0 &= b_0 \left[1 - \frac{m}{b^*} - \frac{3}{2} \left(\frac{m}{b^*}\right) - 4 \left(\frac{m}{b^*}\right) + \dots \right] \\ \text{or } r_0 &= b_0 - \frac{m}{K} - \frac{3}{2} \frac{m^2}{bK^{5/2}} - 4 \frac{m^3}{b^2 K^4} + \dots, \end{aligned} \quad (33)$$

where, we denoted $(1 - a^2/b^2)$ by K .

3.2 Deflection angle of light

As the light is travelling near the gravitating object, the trajectory of the light and the center of mass of the object makes a plane. Since the trajectory of light is in that plane, where $\theta = \pi/2$ as stated earlier, this can be described by the coordinate (r, ϕ) and the expression for the deflection angle of light can be present as below [66, 68]:

$$\Delta\phi = 2 \int_{r_0}^{\infty} \left| \frac{d\phi}{dr} \right| dr - \pi. \quad (34)$$

Now, from Eqs. (10) and (25) we have:

$$\frac{d\phi}{dr} = \frac{1}{r^2 + a^2} \sqrt{\left(\frac{r^2}{r^2 + a^2}\right) / \left(\frac{1}{b^2} - \frac{1 - 2m/r}{r^2 + a^2}\right)}. \quad (35)$$

Therefore, substituting Eq. (35) into Eq. (34) we obtained:

$$\begin{aligned} \Delta\phi &= 2 \int_{r_0}^{\infty} \frac{1}{r^2 + a^2} \sqrt{\left(\frac{r^2}{r^2 + a^2}\right) / \left(\frac{1}{b^2} - \frac{1 - 2m/r}{r^2 + a^2}\right)} dr - \pi \\ &= 2 \int_{r_0}^{\infty} \frac{1}{r^2 + a^2} \sqrt{\left(\frac{r^2}{r^2 + a^2}\right) / \left(\frac{1}{b^2} - \frac{1}{r^2 + a^2} + \frac{2m}{r(r^2 + a^2)}\right)} dr - \pi. \end{aligned} \quad (36)$$

For simplicity, let us assume a variable, $\xi^2 = \frac{r_0^2 + a^2}{r^2 + a^2}$ and a constant, $h = \frac{m}{r_0}$. Then we write the following:

$$\begin{aligned} \xi^2 = \frac{r_0^2 + a^2}{r^2 + a^2} &\Rightarrow r^2 + a^2 = \frac{r_0^2 + a^2}{\xi^2} \\ &\Rightarrow r = \sqrt{\frac{r_0^2 + a^2}{\xi^2} - a^2} \end{aligned} \quad (37)$$

From Eq. (37), we have:

$$dr = -\frac{(r_0^2 + a^2)d\xi}{\xi^2 \sqrt{r_0^2 + a^2 - \xi^2 a^2}}. \quad (38)$$

And now the limits will be as $r \rightarrow \infty$ then $\xi \rightarrow 0$ and as $r \rightarrow r_0$ then $\xi \rightarrow 1$. Substituting from Eqs. (37), (38) into Eq. (36) and simplifying, we have

$$\begin{aligned} \Delta\phi &= -2 \int_1^0 \frac{\xi^2}{(r_0^2 + a^2)} \left[\frac{\frac{r^2 \xi^2}{r_0^2 + a^2}}{\frac{1}{b^2} - \frac{\xi^2}{r_0^2 + a^2} + \frac{2m\xi^2}{r(r_0^2 + a^2)}} \right]^{1/2} \frac{(r_0^2 + a^2)d\xi}{\xi^2 \sqrt{r_0^2 + a^2 - \xi^2 a^2}} - \pi \\ \text{or } \Delta\phi &= 2 \int_0^1 \frac{d\xi}{\sqrt{1 - \frac{2m}{r_0} - \xi^2 + \frac{2m\xi^3}{r_0 \sqrt{1 + \frac{a^2}{r_0^2}(1 - \xi^2)}}}} - \pi \\ &= 2 \int_0^1 \frac{d\xi}{\sqrt{1 - \xi^2 - 2h + 2h\xi^3 P(\xi)}} - \pi, \end{aligned} \quad (39)$$

where, $P(\xi) = \frac{1}{\sqrt{1 + \frac{a^2}{r_0^2}(1 - \xi^2)}}$. If $a = 0$ then, $P(\xi) = 1$ and if $a \neq 0$ then,

$P(\xi) \neq 1$. This clearly indicates that, Eq. (39) reduces to the gravitational deflection of light due to spherically symmetric static objects at $a = 0$. Therefore, the function $P(\xi)$ contains the information about the contribution of ellipsoidal shape (or ellipticity) on the gravitational deflection of light. Equation (39) is the general expression for the deflection of light due to ellipsoidal gravitating object.

3.3 Weak-field condition

For weak field conditions, the trajectory of the light is assumed to be far from gravitating object such that $h = m/r_0 \ll 1$. Then, $[1 - \xi^2 - 2h + 2h\xi^3 P(\xi)]^{-1/2} = Q(\xi)$ can be expanded in Taylor series form into the power of h as follows:

$$Q(\xi) = Q(0) + Q'(0)h + Q''(0)\frac{h^2}{2!} + \dots, \quad (40)$$

where, $(')$ denotes the derivative with respect to (h) . Also we note that $0 \leq \frac{a^2}{r_0^2}(1 - \xi^2) < 1$ within the limit $[0, 1]$ as $a < r_0$. Hence the function $P(\xi)$ can be expand in binomial series form as below:

$$\begin{aligned} P(\xi) &= \left[1 + \frac{a^2}{r_0^2}(1 - \xi^2) \right]^{-\frac{1}{2}} \\ &= 1 - \frac{1}{2} \frac{a^2}{r_0^2}(1 - \xi^2) + \frac{3}{8} \left[\frac{a^2}{r_0^2}(1 - \xi^2) \right]^2 - \frac{5}{16} \left[\frac{a^2}{r_0^2}(1 - \xi^2) \right]^3 + \dots \end{aligned} \quad (41)$$

Deflection of Light due to Spheroidal Oblate Static Objects

So, we obtain a series expression for deflection angle from Eqs. (39) and (40) as below:

$$\begin{aligned}\Delta\phi &= 2 \int_0^1 \left[Q(0) + Q'(0)h + Q''(0)\frac{h^2}{2!} + \dots \right] d\xi - \pi \\ &= \Delta\phi_0 + \Delta\phi_1 + \Delta\phi_2 + \dots - \pi.\end{aligned}\quad (42)$$

Evaluating Eq. (42) term by term and using Eq. (41) we get:

$$\Delta\phi_0 = 2 \int_0^1 Q(0)d\xi = 2 \int_0^1 \frac{d\xi}{\sqrt{1-\xi^2}} = \pi.\quad (43)$$

Similarly, for the first and second order terms, we have

$$\begin{aligned}\Delta\phi_1 &= 2h \int_0^1 Q'(0)d\xi \\ &= 2h \int_0^1 \frac{d}{dh} [1 - \xi^2 - 2h + 2h\xi^3 P(\xi)]^{-1/2} \Big|_{h=0} d\xi \\ &= 2h \int_0^1 \frac{1 - \xi^3 \left[1 - \frac{a^2}{2r_0^2} (1 - \xi^2) \right]}{(1 - \xi^2)^{3/2}} d\xi \\ &= 2h \left(2 + \frac{a^2}{3r_0^2} \right)\end{aligned}\quad (44)$$

and

$$\begin{aligned}\Delta\phi_2 &= h^2 \int_0^1 Q''(0)d\xi \\ &= h^2 \int_0^1 \frac{d''}{dh} [1 - \xi^2 - 2h + 2h\xi^3 P(\xi)]^{-1/2} \Big|_{h=0} d\xi \\ &= h^2 \left[-4 + \frac{15\pi}{4} + \frac{81a^8\pi}{32768r_0^8} - \frac{45a^6\pi}{2048r_0^6} \right. \\ &\quad \left. + \frac{3a^4(-16+5\pi)}{32r_0^4} + \frac{3a^2(-32+15\pi)}{16r_0^2} \right]\end{aligned}\quad (45)$$

Hence, from Eqs. (42), (43), (44) and (45), we have obtained an expression for the deflection of light as:

$$\begin{aligned}\Delta\phi &= 2h \left(2 + \frac{a^2}{3r_0^2} \right) + h^2 \left[-4 + \frac{15\pi}{4} + \frac{81a^8\pi}{32768r_0^8} - \frac{45a^6\pi}{2048r_0^6} \right. \\ &\quad \left. + \frac{3a^4(-16+5\pi)}{32r_0^4} + \frac{3a^2(-32+15\pi)}{16r_0^2} \right] + \mathcal{O}(h)^3\end{aligned}\quad (46)$$

This Eq. (46) is the expression for the gravitational deflection of light due to an ellipsoidal oblate object under weak field condition. Here, it is also observed that, when $a = 0$ Eq. (46) reduces to the standard series expression for the deflection angle of light due to Schwarzschild mass as obtained in [20, 43].

4 Discussion of Results

To understand the important contribution of ellipticity of gravitating object on the deflection of light ray, we consider Sun as our test object. The Sun has Schwarzschild radius approximately $2m = r_g = 3$ km, rotation parameter $\alpha_{\text{rot}} = 1.69749$ km and linear eccentricity $a = 2951.6$ km. Assuming the closest approach of light ray equal to its equatorial radius $R = 695700$ km, we calculated the gravitational deflection angle of light for rotating mass both in prograde and retrograde direction. For that calculation, we used the expression obtained by Chakraborty and Sen [41] considering first-order term and putting charge parameter as zero. Then, we also calculated the angle of deflection due to ellipsoidal objects by using our Eq. (46) and compared both the values with the standard Schwarzschild gravitational deflection. From that, it is observed that the angle of deflection due to rotating mass is slightly greater in the retrograde direction and smaller in the prograde direction compared to Schwarzschild deflection. Again, we also observed that, the deflection angle due to Sun as an oblate object is larger than the Schwarzschild deflection as well as Kerr deflection (shown in Table 1).

Table 1. Comparison of $\Delta\phi$ for Sun in different geometries

No	Geometry	$\Delta\phi$ (arc sec.)	$\Delta\phi - \Delta\phi_s$
1	Kerr (prograde)	1.7789092	-2.411550×10^{-6}
2	Kerr (retrograde)	1.7789140	2.411603×10^{-6}
3	Ellipsoid	1.7789169	5.336716×10^{-6}

Note: $\Delta\phi_s = 1.7789116$ arc sec. (Schwarzschild light deflection angle)

From Table 1, we can see that the contribution of oblateness of the Sun in the deflection of light is sufficiently larger than that due to the rotation parameter. This result showing the contribution of the oblateness (or ellipticity in general) of a gravitating object in the trajectory of light ray tells us clearly that, this is an important and sizeable contribution. So it should be included for all precise calculation of gravitational deflection in future. Again, it is also an important observation that an object of the same mass can have different values of the deflection angle, which means the difference in space-time geometry, according to its state of motion (i.e. static or non-static) and its shape. Here, the oblateness parameter (a) acts as a characteristic factor in Eq. (46) similar to what a quadrupole parameter do as discussed in [51, 56, 57], but they are completely different. On the other hand, it is to be mentioned that the expression obtained

Deflection of Light due to Spheroidal Oblate Static Objects

for the deflection angle of light, Eq. (46) is only for weak-field conditions and the trajectories on the plane of the ellipsoid.

5 Conclusions

In this work, we have derived an expression for deflection of light due to static oblate mass in a converging series form. The expression reduces to Schwarzschild deflection if the ellipticity parameter is set to zero. After calculating the numerical values of deflection for Sun considering separately its rotation and oblateness, we find the contribution due to oblateness is greater than that due to the rotation. Therefore, the contribution of oblateness can't be neglected to have accurate values of deflection in all such future calculations.

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Deflection of Light due to Spheroidal Oblate Static Objects

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