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The Effects of Noncommutativity on the Energy Spectra for Mass-Dependent Klein-Gordon and Shrödinger Equations with Vector Quark-Antiquark Interaction and Harmonic Oscillator Potential: Application to HLM Systems

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Abstract. The analytical solutions of the mass-dependent Klien-Gordon equation (KGE) with the improved harmonic oscillator potential and the improved vector quark-antiquark interaction are derived from the symmetries of three-dimensional relativistic noncommutative quantum mechanics (3D-RNCQM symmetries) using Bopp's shift method and perturbation theory. The energy state equations are sensitive to the global parameters characterizing the noncommutativity space-space (Θ, σ) and the potential parameter (E_{nl}, a,b, c, m_0, k) in addition to the discrete atomic quantum numbers (j, l, s, m). The energy spectra for the mass-dependent Shrödinger equation with improved vector quark-antiquark interaction and harmonic oscillator potential were found using nonrelativistic limit principles. We also used the current findings to determine the heavy-meson masses of charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ in both ordinary nonrelativistic quantum mechanics symmetries and three-dimensional nonrelativistic noncommutative 3D-NRNCQM symmetries.

KEY WORDS: Klein-Gordon equation; Shrödinger equation; Harmonic oscillator potential; vector quark-antiquark interaction; Noncommutative quantum mechanics; star products.

1 Introduction

In the last few decades, the two relativistic Dirac and Klein-Gordon oscillators (DO and KGO) have received much interest. The vast applications of these systems to multiple physical fields, such as the vibrational spectrum of diatomic molecules, have attracted a lot of attention [1], the binding of heavy quarks system [2, 3], and oscillations of atoms in crystal lattices [4]. Bakke and Furtado examined the effect of a Coulomb-type potential on the KGO and found relativistic bound state solutions for both attractive and repulsive Coulomb-type potentials [5]. Vitória and Bakke [6] investigated the behavior of the KGO under the influence of linear and Coulomb-type potentials. Vitória *et al.* investigated the relativistic quantum dynamics of an electrically charged particle subject to the KGO and the Coulomb potential and analyzed the behavior of a relativistic position-dependent mass particle subject to the KGO and the Coulomb potential [7]. Within the Kaluza-Klein theory, Ahmed investigated the generalized KGO with interactions on a curved background [8]. Boumali and Messai studied the influence of the gravitational fields produced by a topology such as cosmic string space-time on a KGO in the presence of a uniform magnetic field [9]. Bahar and Yasuk used the asymptotic iteration method to get the energy eigenvalues and related eigenfunctions for relativistic spin-1particles in the Duffin-Kemmer-Petiau equation with position-dependent mass under equal vector and scalar Coulomb contact [10]. By using the asymptotic iteration and wave function Ansatz method, Bahar and Yasuk [11] presented exact solutions of the mass-dependent Klein-Gordon equation MDKGE for the harmonic oscillator potential (HOP) and the vector quark-antiquark interaction (MD-HOP-VQI), which are considered a spherically symmetrical potential model and present a good description of heavy quarkonioum mass spectra such as charmonium $c\bar{c}$ and bottomonium $b\bar{b}$. The quark-antiquark interaction potential consists of a HOP and Cornell potential (linear and Coulomb potential), which are considered confined terms. The main goal of this research is to enlarge on Bahar and Yasuk's previous work within the MDKGE framework, but in the context of relativistic and nonrelativistic quantum mechanics RNCQM and NRNCQM symmetries, to gain more microscopic scale investigation and scientific knowledge of elementary particles on the nano-scale. In the context of RNCQM and NRNCQM symmetries, the relativistic and nonrelativistic energy levels under HOP and quark-antiquark interaction in the framework of MDKGE and MDSE have yet to be established. Despite the great successes of relativistic quantum mechanics RQM and nonrelativistic quantum mechanics NRQM symmetries known in the literature, many problems remain without a physical solution, such as the non-renormalizable of electroweak interaction, quantum gravity and string theory (see refs. [12–19]). RNCQM and NRNCQM have attracted the attention of many physical researchers. To overcome these numerous challenges, physicists considered expanding the known symmetries of quantum mechanics, which are based on $\left[x_{nc\mu}^{(\mathrm{S,H,I})}, p_{nc\nu}^{(\mathrm{S,H,I})}\right] \neq 0$, to include new postulates (see Refs. [20–29]):

$$\begin{aligned}
\left[\widehat{x}_{\mu}^{\mathrm{S}} \overset{*}{,} \widehat{p}_{\nu}^{\mathrm{S}}\right] &= \left[\widehat{x}_{\mu}^{\mathrm{H}}(t) \overset{*}{,} \widehat{p}_{\nu}^{\mathrm{H}}(t)\right] = \left[\widehat{x}_{\mu}^{\mathrm{I}}(t) \overset{*}{,} \widehat{p}_{\nu}^{\mathrm{I}}(t)\right] = i\hbar_{\mathrm{eff}}\delta_{\mu\nu} \\
\left[\widehat{x}_{\mu}^{\mathrm{S}} \overset{*}{,} \widehat{x}_{\nu}^{\mathrm{S}}\right] &= \left[\widehat{x}_{\mu}^{\mathrm{H}}(t) \overset{*}{,} \widehat{x}_{\nu}^{\mathrm{H}}(t)\right] = \left[\widehat{x}_{\mu}^{\mathrm{I}}(t) \overset{*}{,} \widehat{x}_{\nu}^{\mathrm{I}}(t)\right] = i\theta_{\mu\nu}
\end{aligned} \tag{1}$$

The sum indices (μ, ν) can be equal $(1, 2, 3)$. This algebra presents the novel covariant noncommutative canonical commutations relations (CNCCRs) in RNCQM and NRNCQM symmetries which are known by the noncommutativity of the space-space on the deformed Heisenberg-Weyl algebra. The notations (S, H, and I) denote the Schrödinger, Heisenberg, and interaction pictures (SP, HP, and IP), respectively. The Schrödinger representation has been discussed in the literature, so we have generalized this algebra to include both HP and IP. Here $\hbar_{\text{eff}}$ is the effective Planck constant, $\theta^{\mu\nu} = \epsilon^{\mu\nu}\theta$ ($\epsilon^{\mu\nu}$ is antisymmetric real constant (3×3) matrices and θ is the noncommutative parameter, which is infinitesimal parameters if it is compared with the energy values) and $\delta_{\mu\nu}$ is the identity matrix. The symbol $(*)$ represents the Weyl Moyal star product, which is generalized between two ordinary functions $f(x)$ and $g(x)$ to the new deformed form $\widehat{f}(\widehat{r})\widehat{g}(\widehat{r})$, which can be expressed on the Weyl Moyal star product $f(x) * g(x)$ in RNCQM and NRNCQM symmetries as follows (see refs. [30–35](in our calculation, we have used natural units $c = \hbar = 1$).

$$\begin{aligned} f(x) * g(x) &= \exp\left(i\epsilon^{\mu\nu}\theta\partial_\mu^x\partial_\mu^x\right)(fg)(x) \\ &\approx (fh)(x) - \frac{i\epsilon^{\mu\nu}\theta}{2}\partial_\mu^x f\partial_\mu^x g\rfloor_{x^\mu = x^\nu} + O\left(\theta^2\right) \end{aligned} \tag{2}$$

The idea of including RNCQM and NRNCQM symmetries is not new, as it dates back to the early years of the emergence of quantum mechanics known in the literature. It was proposed by Snyder [36, 37] in 1947 and its geometric analysis was introduced by Connes [38–43]. We hope to find new applications and profound physical interpretations by using a new, updated model of the MD-IHOP and IVQI, which we call the mass-dependent deformed (KGE/SE) for the improved harmonic oscillator potential (IHOP) model and the improved vector quark-antiquark interaction (IVQI) model:

$$\begin{pmatrix} V_{ho}(\widehat{r}) \\ V_{q\bar{q}}(\widehat{r}) \\ m_{ho}(\widehat{r}) \\ m_{q\bar{q}}(\widehat{r}) \end{pmatrix} = \begin{pmatrix} V_{ho}(r) - \dfrac{\partial V_{ho}(r)}{\partial r} \\ V_{ho}(r) - \dfrac{\partial V_{q\bar{q}}(r)}{\partial r} \\ m_{ho}(r) - \dfrac{\partial m_{ho}(r)}{\partial r} \\ m_{q\bar{q}}(r) - \dfrac{\partial m_{q\bar{q}}(r)}{\partial r} \end{pmatrix} \frac{\mathbf{L}\Theta}{2r} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} O\left(\Theta^2\right). \tag{3}$$

Here $V_{ho}(r)$, $V_{q\bar{q}}(r)$, $m_{ho}(r)$ and $m_{q\bar{q}}(r)$ are given by [11]:

$$\begin{cases} V_{ho}(r) = \frac{1}{2}m(r)\,\omega^2 r^2 \\ m_{ho}(r) = m_0 + \frac{1}{2}kr^2 \end{cases} \quad \text{for the HOP model} \tag{4}$$

and

$$\begin{cases} V_{q\bar{q}}(r) = ar^2 + br - c/r \\ m_{q\bar{q}}(r) = m_0 + ar^2 + br - c/r \end{cases} \quad \text{for the IVQP model,} \tag{5}$$

where (a, b, c) are constants while $\omega = \sqrt{k/m(r)}$ is the angular frequency and k is the elastic coefficient. In 3D-RNCQM symmetries the modified effective mass Klein-Gordon equation $(m_{ho}(\widehat{r}), m_{q\overline{q}}(\widehat{r}))$ and the corresponding deformed potentials $(V_{ho}(\widehat{r}), V_{q\overline{q}}(\widehat{r}))$ can be unified in this formula:

$$F(r) \rightarrow F(\widehat{r}) = F(r) - \frac{\partial F(r)}{\partial r} \frac{\mathbf{L}\Theta}{2r} + O\left(\Theta^2\right) \tag{6}$$

It is important to note that Benzair *et al.* used the supersymmetric path integral formalism to study the dynamics of the KGO and the DO in (2+1) dimensions with noncommutativity of the spatial coordinates, calculate the propagator, and derive the energy eigenvalues with their corresponding eigenfunctions [44]. Mirza and Mohadesi also investigated the KGO and DO in a noncommutative space and showed that their behavior is comparable to that of a particle's dynamics in a commutative space and the presence of a constant magnetic field [45]. In this current study, we consider both a study of the mass-dependent KGE with improved vector quark-antiquark interaction and a change in the mass-dependent KGO. The rest of this article is structured as follows: The MD-(HOP and VQI) in RQM is briefly reviewed in the section that follows. Section 3 focuses on the mass-dependent KGE with improved harmonic oscillator potential and improved vector quark-antiquark interaction using the ordinary Bopp's shift method to obtain the effective potentials $V_{\text{eff}}^{ho}(\widehat{r})$ and $V_{\text{eff}}^{q\overline{q}}(\widehat{r})$. Additionally, using perturbation theory, we identify the expectation values of a few radial components in order to determine the energy shift carried on by the effect of the perturbed effective potential of MD-(IHOP and IVQI). The global energy shift and the global energy spectra generated by the IHOP and IVQI models in 3D-RNCQM symmetries are shown in Section 4. The concepts of the nonrelativistic limit, which are well-known in the literature, will be used to determine the eigenvalues of the examined potentials within the context of three-dimensional nonrelativistic NCQM (3D-NRNCQM) symmetries in the next section. In Section 5, we use the results of our research to determine the energy spectra of heavy quarkonium systems, such as charmonium $c\overline{c}$ and bottomonium $b\overline{b}$, which have quark and antiquark flavors with spins-(0 or 1) under MD-IVQI in the 3D-NRNCQM symmetries. The overview and conclusions are reported in Section 6.

2 Revised the MD-(IHOP and IVQI) Models in RQM Symmetries

As already mentioned, our objective is to obtain the spectrum produced with the mass-dependent KGE $m(r)$ with improved harmonic oscillator potential and improved vector quark-antiquark interaction MD-(IHOP and IVQI) models in 3D-RNCQM and 3D-NCQM symmetries. We need to revise the corresponding mass-dependent KGE with HOP and VQI in symmetries of ordinary relativistic quantum mechanics RQM. The 3-dimensional relativistic Klein-Gordon equation RKGE with MD-IHOP and IVQI is given by

$$\left(-\Delta+\left(m\left(r\right)+S_{ho}\left(r\right)^{2}\right)-\left(E_{nl}-V_{ho}\left(r\right)\right)^{2}\right)\Psi\left(r,\theta,\varphi\right)=0 \tag{7}$$

and

$$\left(-\Delta+\left(m\left(r\right)+S_{q\bar{q}}\left(r\right)^{2}\right)-\left(E_{nl}-V_{q\bar{q}}\left(r\right)\right)^{2}\right)\Psi\left(r,\theta,\varphi\right)=0. \tag{8}$$

The two vector potentials $V_{ho}\left(r\right)/V_{q\bar{q}}(r)$ due to the four-vector linear momentum operator $A^{\mu}(V_{ho}\left(r\right)/V_{q\bar{q}}(r),\ \vec{A}\ =\ 0)$ and the space-time scalar potential $S_{ho}\left(r\right)/V_{q\bar{q}}(r)$, E_{nl} represent the relativistic energy eigenvalues in three dimensions while (n and l) represent the principal and orbital quantum numbers, respectively. Since the VQI and HOP have spherical symmetry, allowing the solutions of the time-independent KGE of the known form $\Psi\left(r,\theta,\varphi\right)=\frac{u_{nl}(r)}{r}Y_{l}^{m}\left(\theta,\varphi\right)$ to separate the radial $u_{nl}\left(r\right)$ and angular $Y_{l}^{m}\left(\theta,\varphi\right)$ parts of the wave function and Δ is the ordinary 3-dimensional Laplacian operator. Using the shorthand notation $\zeta_0=E_{nl}^2-m_0^2$, $\zeta_1=k\left(E_{nl}-m_0\right)$, $\epsilon=E_{nl}^2-m_0^2$, $\epsilon_0=2a\left(E_{nl}+m_0\right)$, $\epsilon_1=2b\left(E_{nl}+m_0\right)$ and $\epsilon_2=2c\left(E_{nl}+m_0\right)$. In the presence of vector potentials ($V_{ho}(r)\neq 0$, $V_{q\bar{q}}(r)\neq 0$), $u_{nl}\left(r\right)$ satisfying the following second-order Schrödinger-like equations:

$$\left(\frac{d^2}{dr^2}-\left(E_{\text{eff}}^{ha}+V_{\text{eff}}^{ho}\left(r\right)\right)\right)u_{nl}\left(r\right)=0 \tag{9}$$

$$\left(\frac{d^2}{dr^2}-\left(E_{\text{eff}}^{q\bar{q}}+V_{\text{eff}}^{q\bar{q}}\left(r\right)\right)\right)u_{nl}\left(r\right)=0 \tag{10}$$

with

$$V_{\text{eff}}^{ho}\left(r\right)=\zeta_1 r^2+\frac{l(l+1)}{r^2}\,, \tag{11}$$

$$V_{\text{eff}}^{q\bar{q}}\left(r\right)=\epsilon_0 r^2+\epsilon_1 r-\frac{\epsilon_2}{r}+\frac{l(l+1)}{r^2}\,, \tag{12}$$

$$E_{\text{eff}}^{hc}=m_0^2-E_{nl}^2 \text{ and } E_{\text{eff}}^{q\bar{q}}=m_0^2-E_{nl}^2\,. \tag{13}$$

Ref. [11] gives the energy eigenvalues E_{nl} of the KGE with the vector quark-antiquark interaction and HOP, as follows:

1. For any $(n,l,m)^{\text{th}}$ excited state under the HOP model

$$E_{nl}^2=m_0^2+\sqrt{\frac{1}{4}k\left(E_{nl}-m_0\right)}\left(4n+2+\sqrt{1+4l(l+1)}\right). \tag{14}$$

2. For the ground state ($n=0$) and first excited state ($n=1$) under the vector quark-antiquark interaction model

$$2\left(E_{0l}+m_0\right)c=-2\left(l+1\right)\alpha\,, \tag{15}$$

$$2\left(\frac{\epsilon_1}{2\alpha}+\frac{\epsilon_1}{\alpha}(l+1)+\epsilon\right)(l+1)-(\epsilon_2+2\alpha(l+1))(\epsilon_2+2\alpha(l+2))=0\,, \tag{16}$$

while the corresponding total wave function $\Psi\left(r,\theta,\varphi\right)$

$$\begin{aligned}\Psi^{ho}(r,\theta,\varphi)=&N_{nl}\Big(\frac{2+\sqrt{1+4\zeta_2}}{2}\Big)_n\frac{1}{r}r^{1/2(1+\sqrt{1+4\zeta_2})}\exp\Big(-r^2\frac{\sqrt{\zeta_1}}{2}\Big)\\&\times{}_1F_1\Big(-n,\frac{2+\sqrt{1+4\zeta_2}}{2};\sqrt{\zeta_1}r^2\Big)Y_l^m(\theta,\varphi)\end{aligned}\tag{17}$$

and

$$\begin{aligned}\Psi^0_{q\overline{q}}(r,\theta,\varphi)&=\frac{a_0}{r}\exp\left(-\beta r-\gamma r^2\right)r^{(l+1)}Y_l^m\left(\theta,\varphi\right)\\\Psi^1_{q\overline{q}}(r,\theta,\varphi)&=\frac{(a_0+a_1r)}{r}\exp\left(-\beta r-\gamma r^2\right)r^{(l+1)}Y_l^m\left(\theta,\varphi\right)\\\Psi^n_{q\overline{q}}(r,\theta,\varphi)&=\frac{(a_0+...a_pr^n)}{r}\exp\left(-\beta r-\gamma r^2\right)r^{(l+1)}Y_l^m\left(\theta,\varphi\right).\end{aligned}\tag{18}$$

Here

$$\begin{aligned}\alpha&=\left(-\sqrt{\epsilon_0+\epsilon_1}-\sqrt{\epsilon_0-\epsilon_1}\right)/2,\\\beta&=\left(\sqrt{\epsilon_0+\epsilon_1}+\sqrt{\epsilon_0-\epsilon_1}\right)/2,\\\gamma&=\left(\sqrt{\epsilon_0+\epsilon_1}-\sqrt{\epsilon_0-\epsilon_1}\right)/4\end{aligned}$$

while N_{nl} is normalization constant. The confluent hypergeometric function ${}_1F_1\,(-n,\dfrac{2+\sqrt{1+4\zeta_2}}{2};\sqrt{\zeta_1}r^2)$ can be expressed as a function of the generalized Laguerre polynomials $L_n^{\left(\sqrt{1/4+\zeta_2}\right)}\left(\sqrt{\zeta_1}r^2\right)$ as follows [46]:

$$\begin{aligned}&{}_1F_1(-n,\sqrt{1/4+\zeta_2}+1;\sqrt{\zeta_1}r^2)=\\&\qquad=\frac{n!\Gamma(\sqrt{1/4+\zeta_2}+1)}{\Gamma(\sqrt{1/4+\zeta_2}+n+1)}L_n^{(\sqrt{1/4+\zeta_2})}(\sqrt{\zeta_1}r^2)\end{aligned}\tag{19}$$

This allows us to reformulate the wave function $\Psi^{ho}(r,\theta,\varphi)$ in Eq. (17) as follows:

$$\begin{aligned}\Psi^{ho}(r,\theta,\varphi)=&\frac{N^{ho}_{nln}}{r}r^{1/2\left(1+\sqrt{1+4\zeta_2}\right)}\exp\left(-r^2\frac{\sqrt{\zeta_1}}{2}\right)\\&\times L_n^{\left(\sqrt{1/4+\zeta_2}\right)}\left(\sqrt{\zeta_1}r^2\right)Y_l^m\left(\theta,\varphi\right)\end{aligned}\tag{20}$$

with

$$N^{ho}_{nl}=N_{nl}\frac{n!\Gamma\left(\sqrt{1/4+\zeta_2}+1\right)}{\Gamma\left(\sqrt{1/4+\zeta_2}+n+1\right)}\left(\frac{2+\sqrt{1+4\zeta_2}}{2}\right)\tag{21}$$

3 Solution of DKGE under MD-(IHOP and IVQI) Models in 3D-RNCQM Symmetry

At the beginning of this section, we shall give and define the formula of MD-(IHOP and IVQI) models in the symmetries of relativistic noncommutative three-dimensional real space 3D-RNCQM. To accomplish this, it is useful to write the DKGE by applying the notion of the Weyl Moyal star product, which was previously seen in Eq. (2), to the differential equation that is satisfied by the radial wave function $u_{nl}(r)$ in Eqs. (9) and (10). As a result, the radial wave function $u_{nl}(r)$ in 3D-RNCQM symmetry becomes

$$\begin{aligned}&\left(\frac{d^2}{dr^2}-\left(E_{\text{eff}}^{ha}+V_{\text{eff}}^{ho}\left(r\right)\right)\right)*u_{nl}\left(r\right)=0,\\&\left(\frac{d^2}{dr^2}-\left(E_{\text{eff}}^{q\bar{q}}+V_{\text{eff}}^{q\bar{q}}\left(r\right)\right)\right)*u_{nl}\left(r\right)=0.\end{aligned}\tag{22}$$

We briefly outline Bopp's shift method here; the details can be found in [47–50]. The physicist Fritz Bopp was the first to consider pseudo-differential operators obtained from a symbol by the quantization rules $(x,p)\rightarrow\left(x-\frac{i}{2}\frac{\partial}{\partial p},p+\frac{i}{2}\frac{\partial}{\partial x}\right)$ instead of the ordinary correspondence $(x,p)\rightarrow\left(x,\frac{i}{2}\frac{\partial}{\partial x}\right)$. In the physics literature, this is known as Bopp's shift, and this quantization procedure is called Bopp quantization. It is known to the specialists that Bopp's shift method [47–50] has been applied effectively and has succeeded in simplifying the three basic equations deformed Schrödinger equation DSE [51–62], the DKGE [63–74], the deformed Dirac equation DDE [75–80] and the deformed Duffin-Kemmer equation [81–84] with the notion of star product to the SE, the KGE, the DE and the Duffin-Kemmer equation with the notion of ordinary product, respectively. The results of the application of this method were very useful and yielded promising results in many physical and chemical fields. The method reduced DSE, DKGE, and DDE, to the SE, KGE, and DE, respectively, under simultaneous translation in the space phase. The CNCCRs with star product in Eq. (2) become new CNCCRs without the notion of star product as follows (see, e.g., [47–50]):

$$\begin{aligned}\left[\widehat{x}_{\mu}^{\text{S}},\widehat{p}_{\nu}^{\text{S}}\right]&=\left[\widehat{x}_{\mu}^{\text{H}}\left(t\right),\widehat{p}_{\nu}^{\text{H}}\left(t\right)\right]=\left[\widehat{x}_{\mu}^{\text{I}}\left(t\right),\widehat{p}_{\nu}^{\text{I}}\left(t\right)\right]=i\hbar_{\text{eff}}\delta_{\mu\nu}\\\left[\widehat{x}_{\mu}^{\text{S}},\widehat{x}_{\nu}^{\text{S}}\right]&=\left[\widehat{x}_{\mu}^{\text{H}}\left(t\right),\widehat{x}_{\nu}^{\text{H}}\left(t\right)\right]=\left[\widehat{x}_{\mu}^{\text{I}}\left(t\right),\widehat{x}_{\nu}^{\text{I}}\left(t\right)\right]=i\theta_{\mu\nu}\end{aligned}\tag{23}$$

The generalized positions and momentum coordinates $(\widehat{x}_{\mu}^{\text{S}},\,\widehat{p}_{\nu}^{\text{S}})$, $(\widehat{x}_{\mu}^{\text{H}}(t),\,\widehat{p}_{\nu}^{\text{H}}(t))$ and $(\widehat{x}_{\mu}^{\text{I}}(t),\,\widehat{p}_{\nu}^{\text{I}}(t))$ in the 3D-RNCQM and 3D-NRNCQM symmetries are defined in terms of the corresponding coordinates $(x_{\mu}^{\text{S}},\,p_{\nu}^{\text{S}})$, $(x_{\mu}^{\text{H}}(t),\,p_{\nu}^{\text{H}}(t))$ and

$(x_{\mu}^{\mathrm{I}}(t), p_{\nu}^{\mathrm{I}}(t))$ in the RQM symmetries via, respectively [63–74]:

$$\left\{\begin{array}{c}(x_{\mu}^{\mathrm{S}}, p_{\nu}^{\mathrm{S}}) \\ (x_{\mu}^{\mathrm{H}}(t), p_{\nu}^{\mathrm{H}}(t)) \\ (x_{\mu}^{\mathrm{I}}(t), p_{\nu}^{\mathrm{I}}(t))\end{array}\right. \Longrightarrow \left\{\begin{array}{c}(\widehat{x}_{\mu}^{\mathrm{S}}, \widehat{p}_{\nu}^{S}) = (x_{\mu}^{\mathrm{S}} - \frac{\theta^{\mu\nu}}{2} p_{\nu}^{\mathrm{S}}, \widehat{p}_{\nu}^{\mathrm{S}} = p_{\nu}^{\mathrm{S}}) \\ (\widehat{x}_{\mu}^{\mathrm{H}}(t), \widehat{p}_{\nu}^{\mathrm{H}}(t) = (x_{\mu}^{\mathrm{H}} - \frac{\theta^{\mu\nu}}{2} p_{\nu}^{\mathrm{H}}, \widehat{p}_{\nu}^{\mathrm{H}} = p_{\nu}^{\mathrm{H}}) \\ (\widehat{x}_{\mu}^{\mathrm{I}}(t), \widehat{p}_{\nu}^{\mathrm{I}}(t)) = (x_{\mu}^{\mathrm{I}} - \frac{\theta^{\mu\nu}}{2} p_{\nu}^{\mathrm{I}}, \widehat{p}_{\nu}^{\mathrm{I}} = p_{\nu}^{\mathrm{I}})\end{array}\right. \tag{24}$$

This allows us to find the operator $\widehat{r}^2$ equal $r^2 - \mathbf{L}\Theta$. According to the Bopp shift method, Eq. (22) becomes similar to the following like the Shrödinger equation (without the notions of star product):

$$\begin{aligned}&\left(\frac{d^2}{dr^2} - \left(E_{\text{eff}}^{ha} + V_{\text{eff}}^{ho}(\widehat{r})\right)\right) u_{nl}(r) = 0, \\ &\left(\frac{d^2}{dr^2} - \left(E_{\text{eff}}^{q\bar{q}} + V_{\text{eff}}^{q\bar{q}}(\widehat{r})\right)\right) u_{nl}(r) = 0.\end{aligned} \tag{25}$$

The new operators $V_{\text{eff}}^{ho}(\widehat{r})$ and $V_{\text{eff}}^{q\bar{q}}(\widehat{r})$ can be expressed as

$$\begin{aligned}V_{\text{eff}}^{ho}(\widehat{r}) &= V_{\text{eff}}^{ho}(r) - \frac{\partial V_{\text{eff}}^{ho}(r)}{\partial r}\frac{\mathbf{L}\Theta}{2r} + O\left(\Theta^2\right), \\ V_{\text{eff}}^{q\bar{q}}(\widehat{r}) &= V_{\text{eff}}^{q\bar{q}}(r) - \frac{\partial V_{\text{eff}}^{q\bar{q}}(r)}{\partial r}\frac{\mathbf{L}\Theta}{2r} + O\left(\Theta^2\right).\end{aligned} \tag{26}$$

We substitute Eqs. (26) into the differential Eqs. (25), respectively, to find

$$\begin{aligned}&\left(\frac{d^2}{dr^2} - \left(E_{\text{eff}}^{ha} + V_{\text{eff}}^{ha}(r)\right) - V_{\text{pert}}^{ho}(r)\right) u_{nl}(r) = 0, \\ &\left(\frac{d^2}{dr^2} - \left(E_{\text{eff}}^{q\bar{q}} + V_{\text{eff}}^{q\bar{q}}(r)\right) - V_{\text{pert}}^{q\bar{q}}(r)\right) u_{nl}(r) = 0,\end{aligned} \tag{27}$$

where $V_{\text{pert}}^{ho}(r)$ and $V_{\text{pert}}^{q\bar{q}}(r)$ are additive parts produced by deformed space-space topological properties for improved harmonic oscillator potential and improved quark-antiquark interaction, respectively. We can be presented as follows:

$$\begin{aligned}V_{\text{pert}}^{ho}(r) &= -\frac{\partial V_{\text{eff}}^{ha}(r)}{\partial r}\frac{\mathbf{L}\Theta}{2r} + O\left(\Theta^2\right), \\ V_{\text{pert}}^{q\bar{q}}(r) &= -\frac{\partial V_{\text{eff}}^{q\bar{q}}(r)}{\partial r}\frac{\mathbf{L}\Theta}{2r} + O\left(\Theta^2\right).\end{aligned} \tag{28}$$

Thus, after straightforward calculations, we can obtain $V_{\text{pert}}^{hc}(r)$ and $V_{\text{pert}}^{q\bar{q}}(r)$ as follows:

$$\begin{aligned}V_{\text{pert}}^{ho}(r) &= \left(\frac{l(l+1)}{r^4} - \zeta_1\right)\mathbf{L}\Theta + O\left(\Theta^2\right), \\ V_{\text{pert}}^{q\bar{q}}(r) &= \left(\frac{l(l+1)}{r^4} - \epsilon_0 - \frac{\epsilon_1}{2r} - \frac{\epsilon_2}{2r^3}\right)\mathbf{L}\Theta + O\left(\Theta^2\right).\end{aligned} \tag{29}$$

The mass-dependent KGE with the Harmonic oscillator potential and the vector quark-antiquark interaction model are extended by including new terms proportional to the radial terms: $1/r^4$, $1/r$ and $1/r^3$ to become the mass-dependent deformed KGE with the improved harmonic oscillator potential and improved vector quark-antiquark interaction model and in 3D-RNCQM symmetry. The additive two parts $V_{\text{pert}}^{ho}(r)$ and $V_{\text{pert}}^{q\overline{q}}(r)$ are included in the new effective potentials $V_{\text{eff}}^{ho}(\widehat{r})$ and $V_{\text{eff}}^{q\overline{q}}(\widehat{r})$ are also proportional to the infinitesimal vector $\boldsymbol{\Theta}$ which equal $(\Theta_{12}e_x + \Theta_{23}e_y + \Theta_{13}e_z)$. This enables us to consider the additive effective potentials $V_{\text{pert}}^{ho}(r)$ and $V_{\text{pert}}^{q\overline{q}}(r)$ perturbation potentials in comparison to the main potentials (parent potential operators $V_{\text{eff}}^{ho}(r)$ and $V_{\text{eff}}^{q\overline{q}}(r)$ in the symmetries of 3D-RNCQM, i.e. the inequalities $V_{\text{pert}}^{ho}(r) \ll V_{\text{eff}}^{ho}(r)$ and $V_{\text{pert}}^{q\overline{q}}(r) \ll V_{\text{eff}}^{q\overline{q}}(r)$. That is all physical justifications for using time-independent perturbation theory are met. This allows us to give a complete prescription for determining the energy level of the generalized $(n,l,m)^{\text{th}}$ excited states. In the case of 3D-RNCQM symmetry, we find the expectation values of the radial terms ($1/r^4$, $1/r$ and $1/r^3$), while accounting for the unperturbed wave functions, as shown in Eqs. (17) and (18). Thus, for the mass-dependent KGE with the improved harmonic oscillator potential, we obtain the following results:

$$\left\langle \frac{1}{r^4} \right\rangle_{(nlm)}^{ho} = N_{nln}^{ho} \int_0^{+\infty} r^{\sqrt{1+4\zeta_2}-3} \exp\left(-r^2\sqrt{\zeta_1}\right) \times \left[L_n^{\left(\sqrt{1/4+\zeta_2}\right)}\left(\zeta_1 r^2\right)\right]^2 dr \quad (30)$$

We have applied the following property of the spherical harmonics:

$$\int Y_l^m(\theta,\varphi)\, Y_{l'}^{m'}(\theta,\varphi)\, d\theta d\varphi = \delta_{ll'}\delta_{mm'} .$$

It is useful to carry out a transformation of the variable $t = r^2$, so that Eq. (30) becomes as follows:

$$\left\langle \frac{1}{r^4} \right\rangle_{(nlm)}^{ho} = \frac{N_{nln}^{ho}}{2} \int_0^{} +\infty t^{\left(\frac{1}{2}\sqrt{1+4\zeta_2}-2\right)} \exp\left(-\sqrt{\zeta_1} t\right) \times \left[L_n^{\left(\sqrt{1/4+\zeta_2}\right)}\left(\sqrt{\zeta_1} t\right)\right]^2 dt \,. \quad (31)$$

Comparing Eq. (31) with the integral of the form [85]

$$\int_0^{+\infty} t^{\eta-1}\exp(-\omega t)L_m^{\lambda}(\omega t)L_n^{\beta}(\omega t)dt = \frac{\omega^{-\eta}\Gamma(n-\eta+\beta+1)\Gamma(m+\lambda+1)}{m!n!\Gamma(1-\eta+\beta)\Gamma(\lambda+1)} \times {}_3F_2\left(-m,\eta,\eta-\beta;-n+\eta,\lambda+1,1\right) \quad (32)$$

with $\mathrm{Re}\, l\,(\varepsilon) > 0$ and ${}_3F_2\left(-m,\varepsilon,\varepsilon-\beta;-n+\varepsilon,\lambda+1,1\right)$ is obtained from the generalized hypergeometric function ${}_pF_q\left(\alpha_1,\ldots,\alpha_p;\beta^1,\ldots,\beta^q,1\right)$ for $p=3$ and $q=2$ while Γ denoting the usual Gamma function. We obtain the following results:

$$\left\langle \frac{1}{r^4}\right\rangle^{ho}_{(nlm)} = \frac{N^{ho}_{nln}}{2}\frac{\omega^{-\eta}\Gamma\left(n+2\right)\Gamma\left(n+\sqrt{1/4+\zeta_2}+1\right)}{n!^2\Gamma\left(2\right)\Gamma\left(\sqrt{1/4+\zeta_2}+1\right)} \times {}_3F_2\left(-n,\eta,-1;-n+\eta,\sqrt{1/4+\zeta_2}+1,1\right) \quad (33)$$

with $\eta=\sqrt{1/4+\zeta_2}-1$. Now, for the mass-dependent KGE with the improved vector quark-antiquark interaction, in 3D-RNCQM symmetry, to find the mean values of $\langle 1/r^4\rangle^{q\bar{q}}_{(nlm)}$, $\langle 1/r\rangle^{q\bar{q}}_{(nlm)}$ and $\langle 1/r^3\rangle^{q\bar{q}}_{(nlm)}$, we start by calculating the mean values in the ground state $(n=0,l,m)$ and then generalize the result to any $(n,l,m)^{\mathrm{th}}$ excited state:

$$\left\langle \frac{1}{r^4}\right\rangle^{q\bar{q}}_{(0lm)} = a_0^2\int_0^{+\infty}\exp\left(-2\beta r-2\gamma r^2\right)r^{2(l+1)-4}dr, \quad (34)$$

$$\left\langle \frac{1}{r}\right\rangle^{q\bar{q}}_{(0lm)} = a_0^2\int_0^{+\infty}\exp\left(-2\beta r-2\gamma r^2\right)r^{2(l+1)-1}dr, \quad (35)$$

$$\left\langle \frac{1}{r^3}\right\rangle^{q\bar{q}}_{(0lm)} = a_0^2\int_0^{+\infty}\exp\left(-2\beta r-2\gamma r^2\right)r^{2(l+1)-3}dr. \quad (36)$$

Comparing Eqs. (34), (35) and (36) with the integral of the form [86]:

$$\int_0^{+\infty} x^{\nu-1}\exp\left(-\lambda x^2-\gamma x\right)dx = (2\lambda)^{-\frac{\nu}{2}}\Gamma(\nu)\exp\left(\frac{\gamma^2}{8\lambda}\right)D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right) \quad (37)$$

where $D_{-\nu}(\gamma/\sqrt{2\lambda})$ denote the Parabolic cylinder functions. We obtain the following results:

$$\left\langle \frac{1}{r^4}\right\rangle^{q\bar{q}}_{(0lm)} = a_0^2(4\gamma)^{-\frac{2l-1}{2}}\Gamma(2l-1)\exp\left(\frac{\beta^2}{4\gamma}\right)D_{-(2l-1)}\left(\frac{\beta}{\sqrt{\gamma}}\right), \quad (38)$$

$$\left\langle \frac{1}{r} \right\rangle_{(0lm)}^{q\bar{q}} = a_0^2(4\gamma)^{-(l+1)}\Gamma(2l+2)\exp\left(\frac{\beta^2}{4\gamma}\right)D_{-(2l+2)}\left(\frac{\beta}{\sqrt{\gamma}}\right), \quad (39)$$

$$\left\langle \frac{1}{r^3} \right\rangle_{(0lm)}^{q\bar{q}} = a_0^2(4\gamma)^{-l}\Gamma(2l)\exp\left(\frac{\beta^2}{4\gamma}\right)D_{-2l}\left(\frac{\beta}{\sqrt{\gamma}}\right). \quad (40)$$

Now, we generalize Eqs. (38), (39) and (40) to any $(n,l,m)^{\text{th}}$ excited state

$$\left\langle \frac{1}{r^4} \right\rangle_{(nlm)}^{q\bar{q}} = \int_0^{+\infty} (a_0 + ...a_n r^n)^2 \exp\left(-2\beta r - 2\gamma r^2\right) r^{2(l+1)-4} dr, \quad (41)$$

$$\left\langle \frac{1}{r} \right\rangle_{(nlm)}^{q\bar{q}} = \int_0^{+\infty} (a_0 + ...a_n r^n)^2 \exp\left(-2\beta r - 2\gamma r^2\right) r^{2(l+1)-1} dr, \quad (42)$$

$$\left\langle \frac{1}{r^3} \right\rangle_{(nlm)}^{q\bar{q}} = \int_0^{+\infty} (a_0 + ...a_n r^n)^2 \exp\left(-2\beta r - 2\gamma r^2\right) r^{2(l+1)-3} dr. \quad (43)$$

This allows us to find the corresponding values for any excited quantum state $(n,l,m)^{\text{th}}$ under the influence of the reactions derived from the mass-dependent KGE with the improved vector quark-antiquark interaction, in 3D-RNCQM symmetry, by applying the integration referred to earlier in Eq. (41). Our recent paper is divided into two principal parts, the first one is to correspond to replace the coupling of angular momentum operator with noncommutativity coupling $\mathbf{L}\Theta$ by the new equivalent coupling $\Theta\mathbf{LS}$ (with $\Theta = \left(\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2\right)^{1/2}$), we have chosen the vector $\boldsymbol{\Theta}$ parallel to the spin-s of a particle under MD-(IHOP and IVQI) models and then we replace $\Theta\mathbf{LS}$ by $\frac{\Theta}{2}\left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right)$. Furthermore, in quantum mechanics, the operators ($H_{nc-r}^{ho}/H_{nc-r}^{q\bar{q}}$, $\mathbf{J}^2$, $\mathbf{L}^2$,$\mathbf{S}^2$ and J_z) forms a complete set of conserved physics quantities, and the eigenvalues of the operator ($\mathbf{J}^2-\mathbf{L}^2-\mathbf{S}^2$) are equal values $2k\,(j,l,s) = j(j+1)-l(l+1)-s(s+1)$, with $|l-s| \leq j \leq |l+s|$. Consequently, the energy shift $\Delta E_{ho}^{so}(n,j,l,s)$ and $\Delta E_{q\bar{q}}^{so}(n,j,l,s)$ due to the perturbed spin-orbit coupling generated by the effect of the perturbed effective potentials $V_{\text{pert}}^{ho}(r)$ and $V_{\text{pert}}^{q\bar{q}}(r)$ for any $(n,l,m)^{th}$ excited state, respectively, in 3D-RNCQM symmetry, under the mass-dependent KGE with improved harmonic oscillator potential and the improved vector quark-antiquark interaction, respectively, as follows:

$$\Delta E_{ho}^{so}(n,j,l,s) = k\,(j,l,s)\left(l(l+1)\left\langle \frac{1}{r^4} \right\rangle_{(nlm)}^{ho} - \zeta_1\right) \quad (44)$$

and

$$\Delta E_{hc}^{so}(n,j,l,s) = k(j,l,s)\Bigg(l(l+1)\left\langle \frac{1}{r^4} \right\rangle_{(nlm)}^{q\bar{q}} - \epsilon_0 - \frac{\epsilon_1}{2}\left\langle \frac{1}{r} \right\rangle_{(nlm)}^{q\bar{q}} - \frac{\epsilon_2}{2}\left\langle \frac{1}{r^3} \right\rangle_{(nlm)}^{q\bar{q}}\Bigg). \quad (45)$$

Thus, the spin-orbit corrections created with the effect of $V_{\text{pert}}^{ho}(r)$ and $V_{\text{pert}}^{q\overline{q}}(r)$ for the state $(n=0,l,m)^{\text{th}}$, under MD-(IHOP and IVQI) models, respectively, we find in 3D-RNCQM symmetry

$$\Delta E_{ho}^{so}(0,j,l,s)=k(j,l,s)\Big(l(l+1)\Big\langle\frac{1}{r^4}\Big\rangle_{(0lm)}^{ho}-\zeta_1\Big) \tag{46}$$

and

$$\Delta E_{q\overline{q}}^{so}(0,j,l,s)=k(j,l,s)\Big(l(l+1)\Big\langle\frac{1}{r^4}\Big\rangle_{(0lm)}^{q\overline{q}} - \epsilon_0-\frac{\epsilon_1}{2}\Big\langle\frac{1}{r}\Big\rangle_{(0lm)}^{q\overline{q}}-\frac{\epsilon_2}{2}\Big\langle\frac{1}{r^3}\Big\rangle_{(0lm)}^{q\overline{q}}\Big)\,. \tag{47}$$

The second main part is corresponding to replacing both ($\mathbf{L}\Theta$ and Θ_{12}) by ($\sigma\aleph L_z$ and $\sigma\aleph$), respectively, here $\aleph$ and σ are, respectively symbolize the intensity of the magnetic field induced by the effect of the deformation of space-space geometry and a new infinitesimal noncommutativity parameter, so that the physical unit of the original noncommutativity parameter Θ_{12} (length)^2 is the same unit of $\sigma\aleph$, we have also need to apply $\langle n',l',m'|\,L_z\,|n,l,m\rangle=\hbar m\delta_{nn'}\delta_{ll'}\delta_{mm'}$ (with $-(l,l')\leqslant(m,m')\leqslant+(l,l')$). All of this data allows for the discovery of the new energy shift $\Delta E_{ho}^{m}(n,j,l,m)$ and $\Delta E_{q\overline{q}}^{m}(n,j,l,m)$ due to the modified perturbed Zeeman effect which was generated by the influence of the perturbed effective potentials $V_{\text{pert}}^{ho}(r)$ and $V_{\text{pert}}^{q\overline{q}}(r)$ for any $(n,l,m)^{\text{th}}$ excited state, respectively, in 3D-RNCQM symmetry, under MD-(IHOP and IVQI) models, respectively, as follows:

$$\Delta E_{ho}^{m}(n,j,l,m)=\aleph\Big(l(l+1)\Big\langle\frac{1}{r^4}\Big\rangle_{(nlm)}^{ho}-\zeta_1\Big)\sigma m \tag{48}$$

and

$$\Delta E_{q\overline{q}}^{m}(n,j,l,m)=\aleph\Big(l(l+1)\Big\langle\frac{1}{r^4}\Big\rangle_{(nlm)}^{q\overline{q}} - \epsilon_0-\frac{\epsilon_1}{2}\Big\langle\frac{1}{r}\Big\rangle_{(nlm)}^{q\overline{q}}-\frac{\epsilon_2}{2}\Big\langle\frac{1}{r^3}\Big\rangle_{(nlm)}^{q\overline{q}}\Big)\sigma m\,. \tag{49}$$

Thus, the magnetic corrections created with the effect of $V_{\text{pert}}^{ho}(r)$ and $V_{\text{pert}}^{q\overline{q}}(r)$ for the state $(n=0,l,m)^{\text{th}}$, under the MD-(IHOP and IVQI) models, respectively, in 3D-RNCQM symmetry

$$\Delta E_{ho}^{m}(0,j,l,m)=\aleph\Big(l(l+1)\Big\langle\frac{1}{r^4}\Big\rangle_{(0lm)}^{ho}-\zeta_1\Big)\sigma m \tag{50}$$

and

$$\Delta E_{q\overline{q}}^{m}(0,j,l,m)=\aleph\Big(l(l+1)\Big\langle\frac{1}{r^4}\Big\rangle_{(0lm)}^{q\overline{q}} - \epsilon_0-\frac{\epsilon_1}{2}\Big\langle\frac{1}{r}\Big\rangle_{(0lm)}^{q\overline{q}}-\frac{\epsilon_2}{2}\Big\langle\frac{1}{r^3}\Big\rangle_{(0lm)}^{q\overline{q}}\Big)\sigma m\,. \tag{51}$$

4 Relativistic Results

In this part, we report our results based on the superposition principle which permitted us to deduce the additive energy shift $\Delta E_{ho}^{\text{tot}}(n,j,l,s,m)$ and $\Delta E_{q\overline{q}}^{\text{tot}}(n,j,l,m)$ due to the spin-orbital coupling and modified Zeeman effect induced by the perturbed potentials $V_{\text{pert}}^{ho}(r)$ and $V_{\text{pert}}^{q\overline{q}}(r)$ for any $(n,l,m)^{\text{th}}$ excited state under the mass-dependent KGE with improved Harmonic oscillator potential and the improved vector quark-antiquark interaction, respectively, in 3D-RNCQM symmetry as follows:

$$\Delta E_{ho}^{\text{tot}}(n,j,l,s,m) = \Big(l(l+1)\Big\langle \frac{1}{r^4}\Big\rangle_{(nlm)}^{ho} - \zeta_1\Big)(k(j,l,s)\Theta + \aleph\sigma m) \quad (52)$$

and

$$\Delta E_{q\overline{q}}^{\text{tot}}(n,j,l,m) = \Big(l(l+1)\Big\langle \frac{1}{r^4}\Big\rangle_{(nlm)}^{q\overline{q}} - \epsilon_0 - \frac{\epsilon_1}{2}\Big\langle \frac{1}{r}\Big\rangle_{(nlm)}^{q\overline{q}} - \frac{\epsilon_2}{2}\Big\langle \frac{1}{r^3}\Big\rangle_{(nlm)}^{q\overline{q}}\Big)(k(j,l,s)\Theta + \aleph\sigma m)\,. \quad (53)$$

Thus, the global corrections created with the effect of $V_{\text{pert}}^{ho}(r)$ and $V_{\text{pert}}^{q\overline{q}}(r)$ for the state $(n=0,l,m)^{\text{th}}$, under the mass-dependent KGE with improved Harmonic oscillator potential and the improved vector quark-antiquark interaction, respectively, we find in 3D-RNCQM symmetry

$$\Delta E_{ho}^{\text{tot}}(n,j,l,s,m) = \Big(l(l+1)\Big\langle \frac{1}{r^4}\Big\rangle_{(0lm)}^{ho} - \zeta_1\Big)(k(j,l,s)\Theta + \aleph\sigma m) \quad (54)$$

and

$$\Delta E_{q\overline{q}}^{\text{tot}}(n,j,l,s,m) = \Big(l(l+1)\Big\langle \frac{1}{r^4}\Big\rangle_{(nlm)}^{q\overline{q}} - \epsilon_0 - \frac{\epsilon_1}{2}\Big\langle \frac{1}{r}\Big\rangle_{(nlm)}^{q\overline{q}} - \frac{\epsilon_2}{2}\Big\langle \frac{1}{r^3}\Big\rangle_{(nlm)}^{q\overline{q}}\Big)(k(j,l,s)\Theta + \aleph\sigma m)\,. \quad (55)$$

The above results present the energy shift, generated with the effect of noncommutativity properties of space-space; it depended explicitly on the noncommutativity parameters (Θ, σ) and the parameters (E_{nl}, a,b, c, m_0, k) of the MD-(IHOP and IVQI) models, in addition to the atomic quantum numbers (n,j,l,s,m). It is should be noted that the obtained effective energy $\Delta E_{ho}^{\text{tot}}(n,j,l,s,m)$ and $\Delta E_{q\overline{q}}^{\text{tot}}(n,j,l,m)$ under MD-(IHOP and IVQI) models have a carried unit of energy because it resulted from the perturbed effective energy ($E_{\text{eff}}^{hc} = m_0^2 - E_{nl}^2$ and $E_{\text{eff}}^{q\overline{q}} = m_0^2 - E_{nl}^2$) combined with the same energy square and the mass square where we have the principle of equivalence between mass and energy at higher energy. This allows us to conclude the energy $E_{r-nc}^{ho}(E_{nl},m_0,k,n,j,l,s,m) \equiv E_{r-nc}^{ho}$

and $E^{q\overline{q}}_{r-nc}(E_{nl}, a, b, c, m_0, n, j, l, s, m) \equiv E^{q\overline{q}}_{r-nc}$, in the symmetries of 3D-RNCQM, corresponding generalized $(n, l, m)^{\text{th}}$ excited states, as a function of the shift energy $[\Delta E^{\text{tot}}_{ho}(n, j, l, s, m)]^{1/2}$, $\left[\Delta E^{\text{tot}}_{q\overline{q}}(n, j, l, s, m)\right]^{1/2}$ and E_{nl} due to the effect of MD-(HOP and VQI) models in RQM, which obtained from Eqs. (14), (15) and (16) as follows:

$$E^{ho}_{r-nc} = E_{nl} + \left[\left(l(l+1)\left\langle\frac{1}{r^4}\right\rangle^{ho}_{(nlm)} - \zeta_1\right)(k(j,l,s)\Theta + \aleph\sigma m)\right]^{1/2} \quad (56)$$

and

$$E^{q\overline{q}}_{r-nc} = E_{nl} + \left[\left(l(l+1)\left\langle\frac{1}{r^4}\right\rangle^{q\overline{q}}_{(nlm)} - \epsilon_0 - \frac{\epsilon_1}{2}\left\langle\frac{1}{r}\right\rangle^{q\overline{q}}_{(nlm)} - \frac{\epsilon_2}{2}\left\langle\frac{1}{r^3}\right\rangle^{q\overline{q}}_{(nlm)}\right)(k(j,l,s)\Theta + \aleph\sigma m)\right]^{1/2}. \quad (57)$$

5 The Mass-Dependent Shrödinger Equation with the MD-(IHOP and IVQI) Models in 3D-NRNCQM Symmetry

To realize a study of the nonrelativistic limit, in three-dimensional nonrelativistic noncommutative 3D-NRNCQM symmetries, the mass-dependent Shrödinger equation with MD-(IHOP and IVQI) models, two steps must be applied. The first step corresponds to the nonrelativistic limit, in the usual three-dimensional nonrelativistic quantum mechanics 3D-NRQM symmetry. This is done by applying the following steps, we replace $E_{nl} + m_0$ and $E_{nl} - m_0$, by $2m_0$ and E^{nr}_{nl}, respectively. After straightforward calculation, for the mass-dependent Shrödinger with an improved harmonic oscillator potential, we can obtain the nonrelativistic energy equation as

$$4m_0^2 E^{nr2}_{nl} - \frac{1}{4}kE_{nl}\left(4n+2+\sqrt{1+4l(l+1)}\right)^2 + \frac{1}{4}km_0\left(4n+2+\sqrt{1+4l(l+1)}\right)^2 = 0 \quad (58)$$

with

$$A = 4m_0^2, \quad B = -\frac{k}{4}\left(4n+2+\sqrt{1+4l(l+1)}\right)^2 \quad \text{and} \quad C = \frac{1}{4}km_0\left(4n+2+\sqrt{1+4l(l+1)}\right)^2 \quad (59)$$

The acceptable physical solution is as follows:

$$E^{nr-ho}_{nl} = \frac{4n+2+\sqrt{1+4l(l+1)}}{8m_0^2}\left[\frac{1}{4}k(4n+2+\sqrt{1+4l(l+1)}) + \sqrt{\frac{k^2}{16}(4n+2+\sqrt{1+4l(l+1)})^2 - 4km_0^3}\right]. \quad (60)$$

We rejected the negative kinetic energy solution because it is not physical in the 3D-NRQM symmetries. For the mass-dependent Shrödinger with vector quark-antiquark interaction, the nonrelativistic quantum energy for $(n = 0, l, m)$:

$$\begin{aligned} 2\left(E_{0l}+m_0\right)c &= -2\left(l+1\right)\left(-\sqrt{\epsilon_0+\epsilon_1}-\sqrt{\epsilon_0-\epsilon_1}\right)/2 \\ \longrightarrow E_{0l} &= \frac{\left(l+1\right)^2}{c^2}\left(a-\sqrt{a^2-b^2}\right)-m_0\,. \end{aligned} \tag{61}$$

Now, the new coefficients ζ_1, ϵ_0, ϵ_1 and ϵ_2 which appeared in the relativistic energy in Eqs. (56) and (57). Under nonrelativistic limit will be

$$\begin{aligned} &\alpha \to -2\sqrt{m_0\left(a+b\right)}-2\sqrt{m_0\left(a-b\right)}, \\ &\zeta_1 = k\left(E_{nl}-m_0\right) \to \zeta_1^{nr} = kE_{nl}^{nr}, \\ &\epsilon_0 = 2a\left(E_{nl}+m_0\right) \to \epsilon_0^{nr} = 4am_0, \\ &\epsilon_1 = 2b\left(E_{nl}+m_0\right) \to \epsilon_1^{nr} = 4bm_0, \\ &\epsilon_2 = 2c\left(E_{nl}+m_0\right) \to \epsilon_2^{nr} = 4cm_0. \end{aligned} \tag{62}$$

After a straightforward calculation, the mass-dependent Shrödinger with vector quark-antiquark interaction, the NRQ energy for $(n = 1, l, m)^{\text{th}}$ stat as follows:

$$E_{nl}^{1r} = \left(\frac{c}{l+1}+\frac{\alpha}{2m_0}\right)\left(4cm_0+2\alpha(l+2)\right)-\frac{b}{\alpha}\left(1+2(l+1)\right)(l+1). \tag{63}$$

As a direct consequence, the new relativistic energy in Eqs. (56) and (57) in 3D-RNCQM symmetry under the mass-dependent KGE with improved vector quark-antiquark interaction and improved harmonic oscillator potential will be reduced to corresponding new NR energy in 3D-NRNCQM symmetries under the mass-dependent SE with improved harmonic oscillator potential and improved vector quark-antiquark interaction

1. For any $(n, l, m)^{\text{th}}$ excited state under the HOP model

$$\begin{aligned} E_{nr}^{ho} &= \frac{4n+2+\sqrt{1+4l(l+1)}}{8m_0^2} \\ &\times\left[\frac{1}{4}k(4n+2+\sqrt{1+4l(l+1)})+\sqrt{\frac{k^2}{16}(4n+2+\sqrt{1+4l(l+1)})^2-4km_0^3}\right] \\ &\quad+\left[\left(l(l+1)\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{ho}-kE_{nl}^{nr}\right)\left(k(j,l,s)\Theta+\aleph\sigma m\right)\right]^{1/2}; \end{aligned} \tag{64}$$

2. For the ground state $(n = 0)$ and first excited state $(n = 1)$ under the improved vector quark-antiquark interaction model

$$E_{0r}^{q\overline{q}} = \Big(\frac{c}{l+1}+\frac{\alpha}{2m_0}\Big)(4cm_0+2\alpha(l+2)) - \frac{b}{\alpha}(1+2(l+1))(l+1) + \Big[\Big(l(l+1)\Big\langle\frac{1}{r^4}\Big\rangle_{(0lm)}^{q\overline{q}} - 4am_0 - 2bm_0\Big\langle\frac{1}{r}\Big\rangle_{(0lm)}^{q\overline{q}} - 2cm_0\Big\langle\frac{1}{r^3}\Big\rangle_{(0lm)}^{q\overline{q}}\Big)(k(j,l,s)\Theta+\aleph\sigma m)\Big]^{1/2} \quad (65)$$

and

$$E_{1r}^{q\overline{q}} = \frac{(l+1)^2}{c^2}(a-\sqrt{a^2-b^2}) - m_0 + \Bigg[\Big(l(l+1)\Big\langle\frac{1}{r^4}\Big\rangle_{(1lm)}^{q\overline{q}} - 4am_0 - 2bm_0\Big\langle\frac{1}{r}\Big\rangle_{(1lm)}^{q\overline{q}} - 2cm_0\Big\langle\frac{1}{r^3}\Big\rangle_{(1lm)}^{q\overline{q}}\Big)(k(j,l,s)\Theta+\aleph\sigma m)\Bigg]^{1/2}. \quad (66)$$

Now, we generalize the ground state energy and first excited state in Eqs. (64) and (66) under the improved vector quark-antiquark interaction model for any $(n,l,m)^{\text{th}}$ excited state

$$E_{nr}^{q\overline{q}} = E_{nl}^{nr} + \Bigg[\Big(l(l+1)\Big\langle\frac{1}{r^4}\Big\rangle_{(nlm)}^{q\overline{q}} - 4am_0 - 2bm_0\Big\langle\frac{1}{r}\Big\rangle_{(nlm)}^{q\overline{q}} - 2cm_0\Big\langle\frac{1}{r^3}\Big\rangle_{(1lm)}^{q\overline{q}}\Big)(k(j,l,s)\Theta+\aleph\sigma m)\Bigg]^{1/2}, \quad (67)$$

where E_{nl}^{nr} is the nonrelativistic energy of the vector quark-antiquark interaction model for any $(n,l,m)^{\text{th}}$ excited state in 3D-NRQM symmetries.

6 Spin-Averaged Mass Spectra of HLM under VQI and IVQI Models in 3D-NRQM and 3D-NRNCQM Symmetries

The quark-antiquark interaction potentials, as mentioned in Ref. [11], are spherically symmetrical and provide a good description of heavy-light mesons HLM such as $c\overline{c}$ and $b\overline{b}$. This would give us a strong incentive to dedicate this section to the purpose to determine the modified spin-averaged mass spectra of HLM such as $c\overline{c}$ and $b\overline{b}$ under the mass-dependent SE with the improved vector quark-antiquark interaction by using the following formula:

$$M_{nl}^{hlm} = m_q + m_{\overline{q}} + E_{nl}^{nr} \rightarrow M_{nc-nl}^{hlm} = m_q + m_{\overline{q}} + \begin{cases} \frac{1}{3}\left(E_{nl}^{nc-u}+E_{nl}^{nc-m}+E_{nl}^{nc-l}\right) & \text{for spin-1} \\ E_{nl}^{nc} & \text{for spin-0} \end{cases} \quad (68)$$

The LHS of Eq. (68) describes spin-averaged mass spectra of HLM in usual QM symmetries [87–89], while the RHS is our self generalization to this formula in nonrelativistic NCQM symmetries, m_q and $m_{\overline{q}}$ are the quark mass and the antiquark mass, M_{nl}^{hlm} is the spin-averaged mass spectra of HLM such as $c\bar{c}$ and $b\bar{b}$ under the mass-dependent SE with the vector quark-antiquark interaction in usual NRQM symmetries, E_{nl}^{nr} is the nonrelativistic energy under the mass-dependent SE with the vector quark-antiquark interaction which determined in by generalizing Eqs. (64) and (66) while (E_{nl}^{nc-u}, E_{nl}^{nc-m}, E_{nl}^{nc-l}) are the modified energies of HLM which have spin-1 while E_{nl}^{nc} is the modified energies of HLM which have spin-0. We need to replace the factor $\Upsilon\,(j,l,s)$ with new generalized values as follows:

$$\Upsilon^{n}\,(j,l,s)=\left[j(j+1)-l(l+1)-s(s+1)\right]/2$$
$$=\begin{cases} l/2 & \text{for } (j=l+1, s=1) \\ -1 & \text{for } (j=l, s=1) \\ (-2l-2)/2 & \text{for } (j=l-1, s=1) \\ 0 & \text{for } (j=l, s=0) \end{cases} \tag{69}$$

Allows us to obtain (E_{nl}^{nc-u}, E_{nl}^{nc-m}, E_{nl}^{nc-l}) and E_{nl}^{nc} of the HLM such as $c\bar{c}$ and $b\bar{b}$ as

1. For $j=l+1$ and $s=1$, E_{nl}^{nc-u} can be expressed by the following formula:

$$\begin{aligned} E_{nl}^{nc-u} =& \left(\frac{c}{l+1}+\frac{\alpha}{2m_0}\right)(4cm_0+2\alpha(l+2)) \\ &-\frac{b}{\alpha}\left(1+2(l+1)\right)(l+1)+\Bigg[\left(l(l+1)\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{q\bar{q}}-4am_0\right. \\ &\left.-2bm_0\left\langle\frac{1}{r}\right\rangle_{(nlm)}^{q\bar{q}}-2cm_0\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{q\bar{q}}\right)\left(\Theta\frac{l}{2}+\aleph\sigma m\right)\Bigg]^{1/2}. \end{aligned} \tag{70}$$

2. For $j=l$ and $s=1$, E_{nl}^{nc-m} can be expressed by the following formula:

$$\begin{aligned} E_{nl}^{nc-m} =& \left(\frac{c}{l+1}+\frac{\alpha}{2m_0}\right)(4cm_0+2\alpha(l+2)) \\ &-\frac{b}{\alpha}\left(1+2(l+1)\right)(l+1)+\Bigg[\left(l(l+1)\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{q\bar{q}}-4am_0\right. \\ &\left.-2bm_0\left\langle\frac{1}{r}\right\rangle_{(nlm)}^{q\bar{q}}-2cm_0\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{q\bar{q}}\right)(-\Theta+\aleph\sigma m)\Bigg]^{1/2}. \end{aligned} \tag{71}$$

3. For $j=l-1$ and $s=1$, E_{nl}^{nc-l} can be expressed by the following formula:

$$
\begin{aligned}
E_{nl}^{nc-l} &= \left(\frac{c}{l+1}+\frac{\alpha}{2m_0}\right)\left(4cm_0+2\alpha(l+2)\right) \\
&\quad -\frac{b}{\alpha}\left(1+2(l+1)\right)(l+1)+\left[\left(l(l+1)\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{q\overline{q}}-4am_0\right.\right. \\
&\quad \left.\left.-2bm_0\left\langle\frac{1}{r}\right\rangle_{(nlm)}^{q\overline{q}}-2cm_0\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{q\overline{q}}\right)\left(-\Theta(l+1)+\aleph\sigma m\right)\right]^{1/2}. \qquad (72)
\end{aligned}
$$

and, for $(j=l,\, s=0)$, we have E_{nl}^{nc} as

$$
\begin{aligned}
E_{nl}^{nc} = E_{nl}^{nr} + \left[\left(l(l+1)\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{q\overline{q}}-4am_0\right.\right. \\
\left.\left.-2bm_0\left\langle\frac{1}{r}\right\rangle_{(nlm)}^{q\overline{q}}-2cm_0\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{q\overline{q}}\right)\aleph\sigma m\right]^{1/2}. \qquad (73)
\end{aligned}
$$

By substituting Eqs. (70), (71) and (72) into Eq. (68), the new mass spectrum of the meson systems in ENRQM symmetries under the mass-dependent SE with the improved vector quark-antiquark interaction for any arbitrary radial and angular momentum quantum numbers becomes

$$
M_{nc-nl}^{hlm} = M_{nl}^{hlm} + \begin{cases} \frac{1}{3}\left(E_{nl}^{nc-u}+E_{nl}^{nc-m}+E_{nl}^{nc-l}\right) & \text{for } s=1 \\ \left[\begin{array}{l}\left(l(l+1)\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{q\overline{q}}\right. \\ -4am_0-2bm_0\left\langle\frac{1}{r}\right\rangle_{(nlm)}^{q\overline{q}} \\ \left.-2cm_0\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{q\overline{q}}\right)\aleph\sigma m\end{array}\right]^{1/2} & \text{for } s=0 \end{cases} \qquad (74)
$$

Thus the spin-averaged mass spectra M_{nl}^{hlm} of HLM such as $c\overline{c}$ and $b\overline{b}$ under the mass-dependent SE with the vector quark-antiquark interaction in usual NRQM symmetries

$$
M_{nl}^{hlm} = m_q + m_{\overline{q}} + E_{nl}^{nr} \qquad (75)
$$

is extended to include δM_{nc-nl}^{hlm} in ENRQM symmetries

$$
\begin{aligned}
\delta M_{nc-nl}^{hlm} &= M_{nc-nl}^{hlm} - M_{nl}^{hlm} \\
&= \begin{cases} \frac{1}{3}\left(E_{nl}^{nc-u}+E_{nl}^{nc-m}+E_{nl}^{nc-l}\right) & \text{for } s=1 \\ \left[\begin{array}{l}\left(l(l+1)\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{q\overline{q}}\right. \\ -4am_0-2bm_0\left\langle\frac{1}{r}\right\rangle_{(nlm)}^{q\overline{q}} \\ \left.-2cm_0\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{q\overline{q}}\right)\aleph\sigma m\end{array}\right]^{1/2} & \text{for } s=0 \end{cases} \qquad (76)
\end{aligned}
$$

Which is sensitive to the atomic quantum numbers (n, j, l, s, m), potential depths (E_{nl}^{nr}, m_0, a, b, c) of the vector quark-antiquark interaction potential, and the noncommutativity parameters (Θ, σ) under the deformed properties of space-space. This allows us to realize logical physical limits

$$\lim_{(\Theta,\sigma)\to(0,0)} M_{nc-nl}^{hlm} = M_{nl}^{hlm} \tag{77}$$

to be achieved. It is worth noting that for the two-simultaneous limits $(\Theta, \sigma) \to (0, 0)$, we recover the energy equations for the mass-dependent KGE with the vector quark-antiquark interaction and harmonic oscillator potential in 3D-RQM symmetries, which is obtained in Ref. [11].

7 Conclusions

In summary, this paper presents an approximate analytical solution of the 3-dimensional deformed Klien-Gordon equation in 3D-RNCQM and 3D-NRNCQM symmetries with improved vector quark-antiquark interaction and improved harmonic oscillator potential MD-(IVQI and IHOP). Under the deformed features of space-space, we found new bound-state energies that appear sensitive to quantum numbers (n, j, l, s, m), potential depths (E_{nl}, m_0, k) and (E_{nl}, m_0, a, b, c) of the examined potentials, and noncommutativity parameters (Θ, σ). Moreover, the nonrelativistic limit of the MD-(IVQI and IHOP) in 3D-NRNCQM symmetries has been investigated. The modified spin-averaged mass spectra of heavy and heavy-light mesons HLM such as $c\bar{c}$ and $b\bar{b}$ in both 3D-NRQM (commutative space) and 3D-NRNCQM symmetries were determined applying our results of the new nonrelativistic energies that represent the binding energy between the quark and anti-quark. It is shown that the MD-(IVQI and IHOP) in a 3D-RNCQM has similar behavior to the dynamics of a particle under the mass-dependent KGE with vector quark-antiquark interaction and harmonic oscillator potential in a 3D-RQM symmetry (commutative space) influenced to the effect of constant magnetic field and a self rotational which can be similar to the behavior of coupling to spin-orbit. As a result, the dynamics of MD-(IVQI and IHOP) models in a 3D-RNCQM symmetry under the deformed Klein-Gordon equation are similar to the dynamics of a particle in a 3D-RQM symmetry under the Duffin-Kemmer equation. It's worth noting that the ordinary physical quantities are recovered in Ref. [11] in all situations to form the two simultaneous limits $(\Theta, \sigma) \to (0, 0)$.

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