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Cosmological Parameters and Stability of Bianchi Type-VIII in Sáez-Ballester Theory of Gravitation

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Abstract. In this paper, we have studied Bianchi type VIII cosmological model in the presence of Sáez-Ballester theory of gravitation with an anisotropic dark matter distribution. Exact solutions of the field equations are obtained for law of variation of Hubble parameter. Also we discussed physical parameters, stability and energy conditions of obtained cosmological model.

KEY WORDS: Bianchi type VIII, Sáez-Ballester, anisotropic dark matter.

1 Introduction

The Einstein's general theory of relativity is one of the most important theory in modern theoretical physics, is used to explore cosmological models. It has the potential to describe a real-life physical phenomenon. Some experts in the field have pointed out that Einstein's general relativity has some conceptual and physical difficulties. For example, Einstein himself pointed out that general relativity does not account satisfactorily for the inertial properties of the matter, i.e. Mach's principle is not supported by general relativity. There is also the singularity problem. Therefore, in an attempt to eliminate some of the conceptual and physical difficulties of general relativity, several theories of gravity have been proposed as alternatives to Einstein's theory of general relativity. In alternative theories, some important scalar tensor theories proposed by Brans and Dicke [1], Nordvedt [2], Wagoner [3], Rose [4], Dun [5], Sáez and Ballester [6], Barber [7], Lau and Prokhovnik [8]. Out of which Sáez-Ballester is one of the most important scalar tensor theory because this theory is capable of addressing the query of missing mass in Friedmann–Roberson–Walker flat universe. In Sáez-Ballester theory, the field equations are given by

$$G_{ij} - \omega\varphi^n \left(\varphi_{,i}\varphi_{,j} - \frac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k}\right) = -8\pi T_{ij} \tag{1}$$

and the scalar field (φ) satisfies the following equation

$$2\,\varphi^n \varphi^{i}_{;i} + n\varphi^{n-1}\varphi_{,k}\varphi^k = 0\,, \tag{2}$$

where $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ is the Einstein tensor, R_{ij} is the Ricci tensor, R is the scalar curvature, n an arbitrary constant, ω is a dimensionless coupling constant and T_{ij} is the matter energy-momentum tensor. Here comma and semicolon denote partial and covariant differentiation respectively. Recently, R.L. Naidu et al. [9], Santhi et al. [10], Mishra et al. [11, 12], Pradhan et al. [13] are some of the authors who investigated cosmological model in Sáez –Ballester theory of gravitation.

Bianchi space-time plays an important role in understanding and describing early stages about the evolution of the universe. In particular, it is very important to study the Bianchi universe of types II, VIII and IX because of familiar solutions such as the FRW universe with positive curvature, de Sitter universe, Taub-NUT solutions, and more corresponding to Bianchi II, VIII and IX space-time V.U.M. Rao et al. [14, 15], Y. Aditya et al. [16], Y.V.S.S. Sanyasiraju et al. [17] investigated Bianchi types II, VIII and IX cosmological model using varies theories.

The discovery of invisible matter called dark matter (DM). Dark matter (DM) played an important role in the formation of galaxies. Many researchers have been theoretically and experimentally attracted to the nature of DM. Nayan Sarkar et al. [18], Piyali Bhar et al. [19] are some of the authors who have obtained cosmological model for an anisotropic DM matter distribution the energy momentum tensor. Also, S. Thirukkanesh et al. [20], L. Herrera [21], Nasr Ahmed et al. [22], Geovanny A. Rave Franco et al. [23], Henning Knutsen [24,25], M.I. Wanas et al. [26] have analyzed stability of cosmological models using various methods.

Motivated by the above discussion and observed facts, in the present paper, we propose to study Bianchi type VIII cosmological model in presence of an anisotropic dark matter (DM) distribution. Our paper is organized as follows. In Section 2: Metric and field equations. Section 3: Solutions of the field equations, Section 4: Physical parameters of the model, Section 5: Stability analysis, Section **??**: Energy conditions. The last Section **??** contains some conclusions.

2 Metric and Field Equations

Consider the Bianchi type VIII space-time in the form

$$ds^2 = dt^2 - R^2(d\theta^2 + \cosh^2\theta d\varphi^2) - S^2(d\phi + \sinh\theta)^2\,, \tag{3}$$

where R, S are functions of the proper time t only.

For an anisotropic DM matter distribution the energy momentum tensor can be written as

$$T_{ij} = (\rho + p_t)\, u_i u_j - p_t g_{ij} + (p_r - p_t)\chi_i \chi_j \,, \tag{4}$$

where $\rho = \rho(r)$, $p_r = p_r(r)$ and $p_t = p_t(r)$ stand for the energy density, radial pressure and transverse pressure of fluid sphere respectively $u^i u_i = -\chi^i \chi_i = 1$ and $u^i \chi_i = 0$, u^i is the four velocity vector of the particles and χ^i is the direction of anisotropy.

From Eq. (4), we have

$$T_1^1 = T_2^2 = -p_t \,, \quad T_3^3 = -p_r \quad \text{and} \quad T_4^4 = \rho \,. \tag{5}$$

Using the equations (1), (2), (4) and (5), the field equations of metric (3) are

$$\frac{R_{44}}{R} + \frac{S_{44}}{S} + \frac{R_4 S_4}{RS} + \frac{1}{4}\frac{S^2}{R^2} - \frac{\omega}{2}\varphi^n \varphi_4^2 = -8\pi p_t \,, \tag{6}$$

$$\frac{2R_{44}}{R} + \frac{R_4^2}{R^2} - \frac{1}{R^2} - \frac{3}{4}\frac{S^2}{R^4} - \frac{\omega}{2}\varphi^n \varphi_4^2 = -8\pi p_r \,, \tag{7}$$

$$\frac{2R_4 S_4}{RS} + \frac{R_4^2}{R^2} - \frac{1}{R^2} - \frac{1}{4}\frac{S^2}{R^4} + \frac{\omega}{2}\varphi^n \varphi_4^2 = 8\pi \rho \,, \tag{8}$$

$$\left(\frac{2R_4}{R} + \frac{S_4}{S}\right)\, \varphi_4 + \varphi_{44} + \frac{n}{2}\frac{\varphi_4^2}{\varphi} = 0 \,, \tag{9}$$

1here the subscript '4' after R, S denotes ordinary differentiation with respect to time t.

The anisotropic factor is defined as $\Delta(r) = p_t(r) - p_r(r)$. The radial and transverse equation of parameters are defined as $\omega_r(r) = p_r(r)/\rho(r)$ and $\omega_t(r) = p_t(r)/\rho(r)$, these two are most important tools in the study of anisotropic matter configuration and they satisfy the condition $0 < \omega_r(r), \omega_t(r) < 1$ for physical matter distribution [27].

For Bianchi type VIII cosmological model, the average scale factor $a(t)$ and spatial volume V are given by

$$a(t) = \left(R^2 S\right)^{\frac{1}{3}} \,, \tag{10}$$

$$V = a^3(t) = \left(R^2 S\right) \,. \tag{11}$$

The mean generalized Hubble parameter H for this model is given by

$$H = \frac{1}{3}\left(\frac{2R_4}{R} + \frac{S_4}{S}\right) \,, \tag{12}$$

where H_1, H_2, H_3 are the directional Hubble parameters defined by

$$H_1 = H_2 = \frac{R_4}{R} \,, \quad H_3 = \frac{S_4}{S} \,. \tag{13}$$

The average anisotropy parameter is

$$A_m = \frac{1}{3} \sum_{i=1}^{3} \left(\frac{\Delta H_i}{H} \right)^2 , \tag{14}$$

where $\Delta H_i = H_i - H$. The expansion scalar θ and shear scalar $\sigma$are given by

$$\theta = 3H = \frac{2R_4}{R} + \frac{S_4}{S} , \tag{15}$$

$$\sigma^2 = \frac{3}{2} A_m H^2 = \frac{1}{3} \left[\frac{R_4}{R} - \frac{S_4}{S} \right]^2 . \tag{16}$$

3 Solutions of Field Equations

The field equations (6)–(9) are four equations in six unknown R, S, φ, P_t, P_r and ρ, which are highly nonlinear differential equation so that we need some extra constraints. We use power law relation between an average scale factor $a(t)$ and the scalar field φ given by $\varphi \propto a^n$, where $n \geq 0$, is any integer:

$$\varphi = k a^n .$$

Without loss of generality we take $k = 1$, so that

$$\varphi = a^n . \tag{17}$$

The shear scalar σ is proportional to expansion scalar θ

$$\sigma = k_1 \theta , \tag{18}$$

where k_1 is the proportionality constant.

Using equations (15), (16) and (18), we get

$$\frac{1}{\sqrt{3}} \left[\frac{R_4}{R} - \frac{S_4}{S} \right] = k_1 \left[\frac{2R_4}{R} + \frac{S_4}{S} \right] . \tag{19}$$

Integrating the above equation, we get

$$R = S^{\frac{1-2\sqrt{3}k_1}{2\sqrt{3}k_1 - 1}} . \tag{20}$$

The average scale factor is

$$a(t) = S^{\frac{2N+1}{3}} , \tag{21}$$

where $N = \dfrac{1 - 2\sqrt{3}k_1}{2\sqrt{3}k_1 - 1}$.

From equation (17) and (21), scalar field φ leads to

$$\varphi = S^{n\left(\frac{2N+1}{3}\right)} . \tag{22}$$

With the help of equations (9), (20) and (22), we get

$$R = N_3 \left(k_2 t + k_3\right)^{\frac{N}{N_1+1}} , \tag{23}$$

$$S = N_2 \left(k_2 t + k_3\right)^{\frac{1}{N_1+1}} \tag{24}$$

and the scalar field is given by

$$\varphi = N_4 \left(k_2 t + k_3\right)^{\frac{n(2N+1)}{3(N_1+1)}} . \tag{25}$$

Using equations (23) and (24), Bianchi type VIII cosmological model in equation (3) takes the form

$$ds^2 = dt^2 - N_3^2 \left(k_2 t + k_3\right)^{\frac{2N}{N_1+1}} \left[d\theta^2 + \cosh^2\theta d\varphi^2\right] - N_2^2 \left(k_2 t + k_3\right)^{\frac{2}{N_1+1}} \left[d\phi + \sinh\theta d\varphi\right]^2 . \tag{26}$$

4 Physical and Some Observational Parameters of the Model

For Bianchi type VIII cosmological model in Sáez-Ballester theory, the average scale factor, the spatial volume is given by

$$a(t) = N_2 N_3^2 \left(k_2 t + k_3\right)^{\frac{2N+1}{3(N_1+1)}} \tag{27}$$

and the spatial volume is

$$V = N_2 N_3^2 \left(k_2 t + k_3\right)^{\frac{2N+1}{N_1+1}} . \tag{28}$$

The scalar expansion θ, the directional Hubble parameter H and shear scalar σ^2 are given by

$$\theta = \frac{(2N+1)k_2}{(N_1+1)(k_2 t + k_3)} , \tag{29}$$

$$H = \frac{(2N+1)k_2}{3\,(N_1+1)(k_2 t + k_3)} , \tag{30}$$

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij} = \frac{(N-1)k_2}{3\,(N_1+1)(k_2 t + k_3)} . \tag{31}$$

The expressions for transverse pressure (p_t), radial pressure (p_r), energy density (ρ), and anisotropic factor (Δr) are obtained in the following forms:

$$p_t = \frac{1}{8\pi(k_2t+k_3)^2}\left\{k_2^2\left(\frac{\omega}{2}N_7(k_2t+k_3)^M - N_5\right) - \frac{1}{4}N_6(k_2t+k_3)^{M_1}\right\}, \tag{32}$$

$$p_r = \frac{1}{8\pi(k_2t+k_3)^2}\left\{\frac{1}{N_3^2}(k_2t+k_3)^{M_2} + \frac{3}{4}N_6(k_2t+k_3)^{M_1} + \frac{\omega}{2}N_7k_2^2(k_2t+k_3)^M - k_2^2N_8\right\}, \tag{33}$$

$$\rho = \frac{1}{8\pi(k_2t+k_3)^2}\left\{k_2^2N_9 + \frac{\omega}{2}N_7k_2^2(k_2t+k_3)^M - \frac{1}{N_3^2}(k_2t+k_3)^{M_2} - \frac{1}{4}N_6(k_2t+k_3)^{M_1}\right\} \tag{34}$$

and the anisotropic factor is

$$\Delta(r) = p_t(r) - p_r(r) = \frac{1}{8\pi(k_2t+k_3)^2}\left\{k_2^2M_3 - \frac{1}{N_2^2}(k_2t+k_3)^{M_2} - N_6(k_2t+k_3)^{M_1}\right\}, \tag{35}$$

where

$$M = \frac{n(n+2)(2N+1)}{3(N_1+1)}, \qquad M_1 = \frac{2N_1 - 4(N-1)}{N_1+1},$$

$$M_2 = \frac{2(N_1-N+1)}{N_1+1}, \qquad M_3 = \frac{(2N-N_1)(N-1)}{(N_1+1)^2}.$$

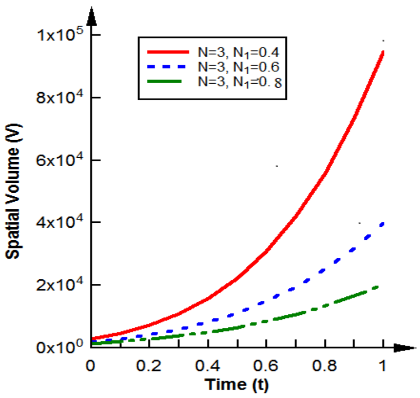

Figure 1. Plot of spatial volume versus time for $k_2 = k_3 = 2$.

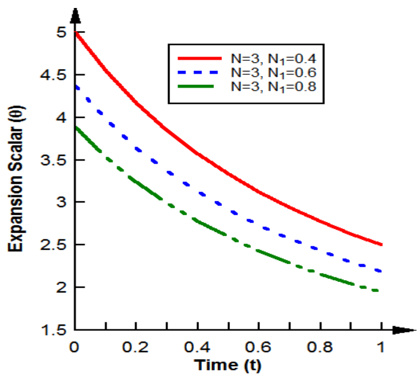

Figure 2. Plot of expansion scalar versus time for $k_2 = k_3 = 2$.

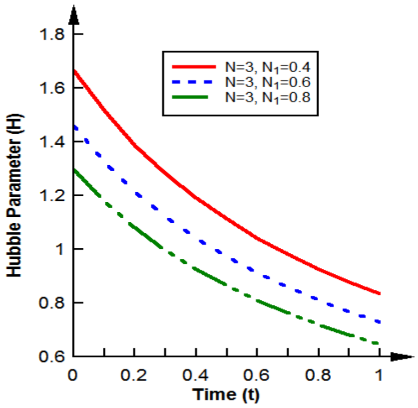

Figure 3. Plot of Hubble parameter versus time for $k_2 = k_3 = 2$.

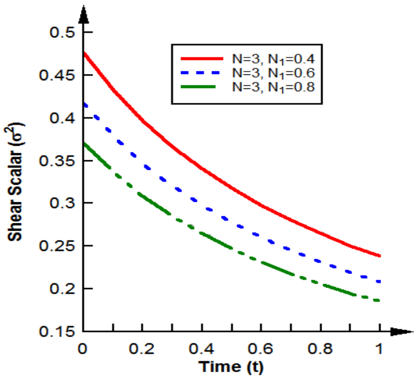

Figure 4. Plot of shear scalar versus time for $k_2 = k_3 = 2$.

The graphs are plotted for particular values of constants and the other integration constants.

5 Stability Analysis

Using different aspects we discuss the stability of the model.

5.1 Speed of sound

The physical quantities $v_r^2 = \dfrac{dp_r(r)}{d\rho(r)}$ and $v_t^2 = \dfrac{dp_t(r)}{d\rho(r)}$ describing the behavior of radial and tangential sound speeds respectively. When both the ratios are positive, i.e. $v_r^2 > 0$ and $v_t^2 > 0$, we have a stable model whereas when both the ratios are negative, i.e. $v_r^2 < 0$ and $v_t^2 < 0$, we have an unstable model.

In these model, it is observed that both the ratios are positive, i.e. $v_r^2 > 0$ and $v_t^2 > 0$. So given cosmological model is stable (see Figures 5 and 6).

5.2 The causality condition

In addition to the positivity of v_r^2 and v_t^2, the causality condition states that the speed of sound is less than the speed of light and can be describe as

$$0 \leq v_r = \sqrt{\frac{dp_r(r)}{d\rho(r)}} \leq 1 \quad \text{and} \quad 0 \leq v_r = \sqrt{\frac{dp_r(r)}{d\rho(r)}} \leq 1\,.$$

In the given cosmological model, the speed of sound is greater than one but the speed of sound is not less than the speed of light, i.e. it does not lie between 0 and 1. So, the causality condition is not satisfied.

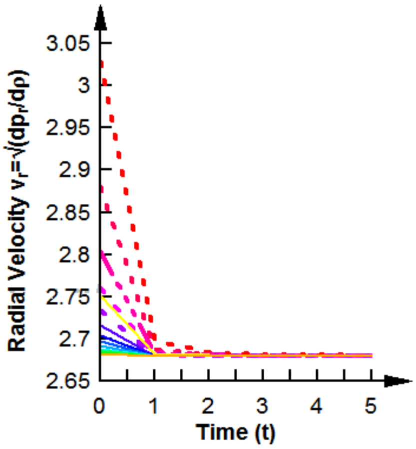

Figure 5. Plot of radial velocity versus time for $k_2 = k_3 = 1.7$.

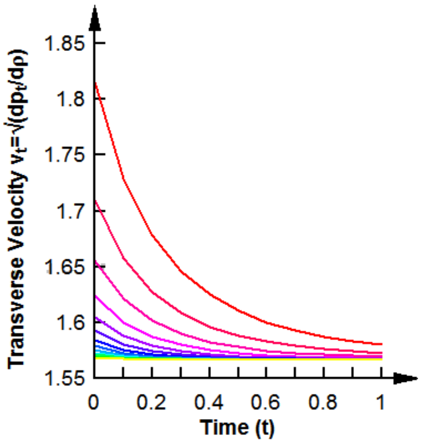

Figure 6. Plot of transverse velocity versus time for $k_2 = k_3 = 16$.

5.3 The stability condition

To study the stability Sarkar et al. [18] and Abreu et al. [28] provides the conditions with respect to the stability factor $\left(v_t^2 - v_r^2\right)$ for an anisotropic fluid model. The stability condition is described as:

The region is potentially stable if $-1 < \left(v_t^2 - v_r^2\right) < 0$ and the region is potentially unstable if $0 < \left(v_t^2 - v_r^2\right) < 1$.

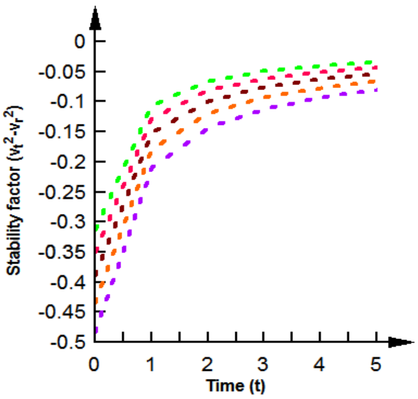

Figure 7. Plot of stability factor versus time for $k_2 = k_3 = 16$.

From Figure 7, it is clear that stability factor $\left(v_t^2 - v_r^2\right)$ lies between -1 and 0. So, the model is potentially stable.

6 Energy Conditions

The physical competency of the model can be checked through testing the energy conditions. The energy conditions are Null, Weak, Strong energy conditions. All these energy conditions can be presented as

(i) Null energy condition (NEC_r) : $\rho(r) - p_r(r) \geq 0$.
(ii) Null energy condition (NEC_t) : $\rho(r) - p_t(r) \geq 0$.
(iii) Weak energy condition (NEC_r) : $\rho(r) \geq 0$, $\rho(r) - p_r(r) \geq 0$.
(iv) Weak energy condition (NEC_t) : $\rho(r) \geq 0$, $\rho(r) - p_t(r) \geq 0$.
(v) Strong energy condition (WEC) : $\rho(r) - p_r(r) - 2p_t(r) \geq 0$.

In these model, it is observed that all the mentioned energy conditions are satisfied. Consequently, our solutions represent the DM configuration, which is also physical in nature.

7 Discussion and Conclusion

In this paper, we have considered Bianchi type VIII cosmological model in Sáez-Ballester theory with an anisotropic dark matter fluid. For solving the field equations, we used the law of Hubble parameter, we use relation between an average scale factor $a(t)$and scalar field φ. Also, the relation between shear scalar σ and expansion scalar θ. We found metric potential, radial pressure, transverse pressure, anisotropic factor and energy density. The values of all quantities are depends upon the values of constants, for particular values of constants radial pressure, transverse pressure, anisotropic factor and energy densities are positive. Graphically, we observe that spatial volume increases as time increases, it means that expansion of the universe start at finite volume and it is expanding as time increases, the expansion scalar, Hubble parameter and shear scalar decreases gradually as time increases, it means all the physical parameters are well behaved. Using different aspects we discuss the stability of the obtained cosmological model. From Figures 5 and 6, it is observed that radial and tangential sound speeds are positive, i.e. $v_r^2 > 0$ and $v_t^2 > 0$ it gives stable cosmological model. In addition to the positivity of v_r^2 and v_t^2, the causality condition states that speed of sound is less than speed of light. In the given cosmological model, speed of sound is greater than one but speed of sound is not less than speed of light, i.e. it does not lie between 0 and 1. So, for obtained cosmological model the causality condition is not satisfied. To study the stability, Sarkar et al. [18] and Abreu et al. [28] provides the conditions with respect to the stability factor $\left(v_t^2 - v_r^2\right)$ for an anisotropic fluid model. From Figure (7), it is clear that stability factor $\left(v_t^2 - v_r^2\right)$ lies between -1 and 0. So the model is potentially stable. Also we obtained different energy conditions and it is observed that all the mentioned energy conditions are satisfied.

References

[1] C.H. Brans, R.H. Dicke (1961) *Phys. Rev. A* **124** 925.

[2] K. Nordverdt (1970) *Astrophys. J.* **161** 1059.

[3] R.V. Wagoner (1970) *Phys. Rev. D* **1** 3209.

[4] D.K. Ross (1972) *Phys. Rev. D* **5** 284.

[5] K.A. Dunn (1974) *J. Math. Phys.* **15** 2229.

[6] D. Sáez, V.J. Ballester (1985) *Phys. Lett. A* **113** 467.

[7] G.A. Barber (1985) *Gen. Rel. Gravit*. **14** 117.

[8] D. La, P.J. Steinhardt (1989) *Phys. Rev. Lett*. **62** 376.

[9] R.L. Naidu, Y. Aditya, K. Deniel Raju, T. Vinutha, D.R.K. Reddy (2021) *New Astronomy* **85** 101564.

[10] M. Vijaya Santhi, Y. Sobhanbabu (2020) *Eur. Phys. J. C* **80** 1198.

[11] R.K. Mishra, Avtar Chand (2020) *Astrophys. Space Sci.* **365** 76.

[12] R.K. Mishra, Heena Dua (2019) *Astrophys. Space Sci.* **364** 195.

[13] Anirudh Pradhan, Ajay Kumar Singh, D.S. Chouhan (2013) *Int. J. Theor. Phys.* **52**(1) 266-278.

[14] V.U.M. Rao, K.V.S. Sireesha (2012) *Int. J. Theor. Phys.* **51** 3013-3020.

[15] V.U.M. Rao, M.V. Santhi (2011) *J. Mod. Phys.* **2** 1222-1228.

[16] Y. Aditya, V.U.M. Rao, M. Vijya Santhi (2016) *Astrophys. Space Sci.* **361** Article number: 56.

[17] Y.V.S.S. Sanyasiraju, V.U.M. Rao (1992) *Astrophys. Space Sci.* **189** 39-43.

[18] Nayan Sarkar, Susmita Sarkar, Ksh. Newton Singh, Farook Rahaman (2020) *Eur. Phys. J. C* **80** 255.

[19] Piyali Bhar, Ksh. Newton Singh, Nayan Sarkar, Farook Rahaman (2017) *Eur. Phys. J. C* **77** 596.

[20] S. Thirukkanesh, Robert S. Bogadi, Megandhren Govender, Sibusiso Moyo (2021) *Eur. Phys. J. C* **81** 62.

[21] L. Herrera (2020): arXiv:2005.06358v1[gr-qc].

[22] Nasr Ahmed, Sultan Z. Alamri (2018) *Res. Astron. Astrophys.* **18** 123.

[23] G.A.R. Franco, C. Escamilla-Rivera, J. Levi Said (2020) *Eur. Phys. J. C* **80** 677.

[24] Henning Knutsen (1988) *Astrophys. Space Sci.* **140** 385-401.

[25] Henning Knutsen (1988) *Astrophys. Space Sci.* **232** 163-174.

[26] M.I. Wanas, M.A. Bakry (1995) *Astrophys. Space Sci.* **228** 239-253.

[27] F. Rahaman, S. Ray, A.K. Jafry, K. Chakraborty (2010) *Phys. Rev. D* **82** 104055.

[28] H. Abreu, H. Hernandez, L.A. Nunez (2007) *Class. Quantum Gravity* **24** 4631.