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Conharmonically Flat Spacetimes in FRW Cosmological Model

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Abstract. The object of the present paper is to investigate conharmonically flat spacetime in the framework of FRW model. In this regard we have considered perfect fluid as a source of matter distribution of the universe. We solved Einstein's field equations with variable cosmological term using special variational law $\Lambda = \beta H^2$, where β is a constant and H is the Hubble's parameter. Moreover, we have described a cosmological scenario of the universe with these solutions and shown that this model is in accordance with the recent day observations.

KEY WORDS: Conharmonically flat, Einstein's field equation, Cosmic snap parameter, De-Sitter universe, FRW model, ΛCDM model.

1 Introduction

For development of a physical theory, we need a model consistent with known facts of a system. Due to development in Cosmology and Di-Sitter space-time, theoreticians are trying to solve Einstein's field equation to predict various phenomena of the universe. The most successful model, having tremendous prediction power for Cosmology, was obtained by Friedman in 1922 as well as by Robertson and Walker in 1930, which is termed as Friedman-Robertson-Walker (FRW) model. This is a non-static symmetric homogeneous model obeying the cosmological principle. This model is so successful that the standard hot big-bang theory is based on it.

Conharmonic transformations are a special type of conformal transformations preserving the harmonicity of functions. It is known that a harmonic function is defined as a function whose Laplacian vanishes. A harmonic function is not invariant under conformal transformation, in general. The condition under which a harmonic function remains invariant has been studied by Ishii [1] who introduced the conharmonic transformation as a subgroup of the conformal transformation $\bar{g_{ij}} = e^{2\sigma} g_{ij}$, where σ is a real function, satisfying the condition

$$\sigma^{i}_{,i} + \sigma_{,i}\sigma^{i}_{,} = 0, \tag{1}$$

where comma denotes the covariant differentiation with respect to the metric g. In 1957, Ishii [1] introduced the notion of conharmonic curvature tensor L^h_{ijk} in an n-dimensional Riemannian space defined by

$$L^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}(R_{ij}\delta^h_k - R_{ik}\delta^h_j + g_{ij}R^h_k - g_{ik}R^h_j), \tag{2}$$

where R^h_{ijk} is the curvature tensor of type (1,3), R_{ij} is the Ricci tensor of type (0,2), R^i_j is the Ricci tensor of type (1,1) given by $R^i_j = g^{hi}R_{hj}$ and g_{ij} is the metric tensor. Here $R_{ij} = R^h_{ijh}$ and the scalar curvature R is given by $R = g^{ij}R_{ij}$. For relativistic four-dimensional spacetime the conharmonic curvature tensor L^h_{ijk} is defined by

$$L^h_{ijk} = R^h_{ijk} - \frac{1}{2}(R_{ij}\delta^h_k - R_{ik}\delta^h_j + g_{ij}R^h_k - g_{ik}R^h_j). \tag{3}$$

Now we suppose that the spacetime is conharmonically flat. Then from (3) we get

$$R^h_{ijk} = \frac{1}{2}(R_{ij}\delta^h_k - R_{ik}\delta^h_j + g_{ij}R^h_k - g_{ik}R^h_j). \tag{4}$$

Contracting h and k in (4) we obtain

$$R_{ij} = \frac{1}{2}(4R_{ij} - R_{ij} + Rg_{ij} - R_{ij}),$$

which implies that

$$R = 0. \tag{5}$$

Several authors [2–6] have studied FRW Cosmological Model in different ways. It is to be noted that Tiwari et al. [5] studied conharmonically flat FRW model. They have proved that conharmonically flat space is Einstein which is not correct. Motivated by above studies in the present paper we characterize conharmonically flat FRW cosmological model.

The present paper is organized as follows: After introduction in Section 2 we study FRW metric and Einstein's field equations associated with special variational law and Friedmann equations. In the next Section 3 we consider the solutions of field equations and interpret the cosmological scenarios of the universe in different cases. We estimated some cosmological parameters in Section 4. Finally, we drawn an overall conclusion regarding conharmonically flat spacetime in the framework of FRW cosmological model with perfect fluid as a source of matter distribution of the universe.

2 Metric and Field Equations

In isotropic and homogeneous FRW model(Robertson 1935, Walker 1936), the metric of the spacetime has the following form:

$$ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2], \tag{6}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, a(t) is the scale factor which describe how the distance between any two world lines changes with cosmic time t, r is the co-moving radial coordinates. It is important to make some remarks on the scale factor a(t) in FRW model given by (6). Its role is very important in the study of cosmic dynamics, in the sense, that its growth with time ensures expansion of the universe. It is obtained by solving Friedmann equation giving cosmological dynamics. Increasing a(t) with time t shows that galaxies, at the t = constant hypersurface, are moving away from each other. Obviously, it is possible when this hypersurface expands. This phenomenon is analogous to moving colored patches on the surface of an expanding balloon when air is pumped into it. The θ and ϕ parameters are the usual azimuthal and polar angles of spherical co-ordinates with $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The coordinates (t, r, θ, ϕ) are called comoving coordinates. Also k is the curvature of the spacetime which take the values -1, 0, 1 representing open, flat and closed universe respectively. The spatial proper volume of this model is $V(t) = 2\Pi^2 a^3(t)$. Moreover, it is to be noted that according to geometrical analysis, we can not say what should be the actual value of k. But we can think of only three possible values of k given above.

With the metric (6), we can set about computing the connection coefficients and curvature tensor. Setting $\dot{a} = \dfrac{da}{dt}$, the Christoffel symbols are given by the following:

$$\Gamma^0_{11} = \frac{a\dot{a}}{1-kr^2},\ \Gamma^1_{11} = \frac{kr}{1-kr^2},\ \Gamma^0_{22} = a\dot{a}r^2,\ \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta,$$

$$\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a},\ \Gamma^1_{22} = -r(1-kr^2),\ \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta,$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r},\ \Gamma^2_{33} = -\sin\theta\cos\theta,\ \Gamma^3_{23} = \cot\theta.$$

The non-zero components of the Ricci tensor are

$$R_{00} = -3\frac{\ddot{a}}{a},\quad R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2},$$

$$R_{22} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k),\quad R_{33} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\sin^2\theta$$

and the Ricci scalar is then

$$R = 6\Big\{\frac{\ddot{a}}{a} + \Big(\frac{\dot{a}}{a}\Big)^2 + \frac{k}{a^2}\Big\}.$$

The expression for Einstein's relativistic field equations($c = 1$) is

$$R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi GT_{ij} + \Lambda g_{ij}. \tag{7}$$

We assume that the energy-momentum tensor of a perfect fluid is given by

$$T_{ij} = (p + \rho)u_i u_j + pg_{ij}. \tag{8}$$

Also we take the equation of state as

$$p = \omega\rho, \tag{9}$$

where $0 \leq \omega \leq 1$, p and ρ are respectively pressure and energy density of the cosmic fluid and u_i is unit flow vector such that $u_i = (1, 0, 0, 0)$.

In view of equation (5), the Einstein's field equations (7) reduce to the form

$$R_{ij} = -8\pi GT_{ij} + \Lambda g_{ij}. \tag{10}$$

For the metric (6) and the energy-momentum tensor (8), the Einstein's field equations (10) reduce to

$$3(\dot{H} + H^2) = 8\pi G\rho + \Lambda, \tag{11}$$

and

$$\dot{H} + 3H^2 + \frac{2k}{a^2} = -8\pi Gp + \Lambda, \tag{12}$$

where $H = \dot{a}/a$ is the Hubble parameter. Here and elsewhere an over-head dot denotes ordinary differentiation with respect to cosmic time t.

For the flat($k = 0$) FRW metric (6), the equation (12) reduces to

$$\dot{H} + 3H^2 = -8\pi Gp + \Lambda. \tag{13}$$

3 Solution of Field Equations

The system of equations (9), (11) and (13) give only three equations in four unknowns H, p, ρ and Λ. Hence one extra equation is needed to solve the system completely. There is significant theoretical evidence for the phenomenological Λ-decay scenarios have been considered by many authors in different ways such as Wang and Meng [7], Ray et al. [8]. Recently, Kumar and Srivastava [2] have studied FRW cosmological model for conharmonically flat spacetime. We now consider the following special laws for the decay of Λ-cosmological term [9, 10]

$$\Lambda = \beta H^2, \tag{14}$$

where β is a constant. Here β represents the ratio between vacuum and critical densities.

From equations (9), (11), (13) and (14), we obtain

$$\dot{H} + \frac{(3-\beta)(1+\omega)}{1+3\omega} H^2 = 0. \tag{15}$$

Integrating (15) with respect to t yields

$$H(t) = \frac{1+3\omega}{\{(3-\beta)(1+\omega)t + c_1(1+3\omega)\}}, \tag{16}$$

where c_1 is a constant of integration.

From equation (16), we can easily obtain the value of the scale factor $a(t)$ as follows:

$$a(t) = c_2\{(3-\beta)(1+\omega)t + c_1(1+3\omega)\}^{\frac{1+3\omega}{(3-\beta)(1+\omega)}}, \tag{17}$$

where c_2 is a constant of integration. Such constant of integration is related to the choice of origin of cosmic time.

Now we analyze scenarios for different values of ω as follows:

3.1 Matter dominated era

For $\omega = 0$, from equation (17) we obtain the scale factor as

$$a(t) = c_2\{(3-\beta)t + c_1\}^{\frac{1}{(3-\beta)}}. \tag{18}$$

In this case the spatial volume V, matter density ρ, isotropic pressure p and cosmological constant Λ are given by

$$V = 2\pi^2 c_2^3\{(3-\beta)t + c_1\}^{\frac{3}{(3-\beta)}}, \tag{19}$$

$$\rho = \frac{1}{8\pi G}\frac{6(3-\beta)}{\{(3-\beta)t + c_1\}^2}, \tag{20}$$

$$p = 0, \tag{21}$$

$$\Lambda = \frac{3\beta}{\{(3-\beta)t + c_1\}^2}. \tag{22}$$

The expansion scalar Θ and density parameter Ω are given by

$$\Theta = \frac{3}{(3-\beta)t + c_1}, \tag{23}$$

and

$$\Omega = 2(3-\beta). \tag{24}$$

The deceleration parameter q is defined as $q = -1 - \frac{\dot{H}}{H^2}$. Thus we have the following

$$q = 2 - \beta. \tag{25}$$

The deceleration parameter q is dimensionless and a negative value of q represents cosmic acceleration where as a positive q gives decelerating universe. Equations (24) and (25) together yield

$$\Omega = 2(q+1). \tag{26}$$

The vacuum energy density ρ_v and the critical energy density ρ_c are given by

$$\rho_v = \frac{3\beta}{8\pi G\{(3-\beta)t + c_1\}^2}, \tag{27}$$

and

$$\rho_c = \frac{3}{8\pi G\{(3-\beta)t + c_1\}^2}. \tag{28}$$

From equations (18) and (19) of the physical parameters of the model, we observe that the scale factor $a(t)$ and the spatial volume V are zero at $t = t_1$, where $t_1 = -c_1/(3-\beta)$. It follows that the universe under the considered model has a finite-time big-bang singularity at $t = t_1$ which shifts to initial singularity by setting $c_1 = 0$. Here the Hubble parameter diverges at the time $t = t_1$ under the assumption $c_1 = 0$. Moreover, the expansion scalar Θ, isotropic pressure p and matter density ρ all diverge at the time $t = t_1$. Also, in the limiting case for large t, all the physical parameters $H(t)$, Θ, p and ρ converge to zero where as the scale factor $a(t)$ and the volume V become infinite. Thus the considered model in this case represents a cosmological scenario in which the universe starts from a finite-time big-bang singular state and expands with cosmic time t. From equation (25) we observed that for $\beta < 2$ the value ofthe deceleration parameter q is positive which indicates the decelerating phase of expansion of the observed universe. This is mainly responsible for structure formation [11]. For $\beta > 2$ the current model indicates the accelerating phase of expansion. Recent observations favor accelerating models but they do not rule out decelerating models which are also consistent with present day observations.

3.2 Stiff fluid era

In this case, from equation (17), the scale factor $a(t)$ becomes

$$a(t) = c_2\{2(3-\beta)t + 4c_1\}^{\frac{2}{(3-\beta)}}. \tag{29}$$

Here spatial volume V, matter density ρ, isotropic pressure p and cosmological constant Λ are as follows:

$$V = 2\pi^2 c_2^3\{2(3-\beta)t + 4c_1\}^{\frac{6}{(3-\beta)}}, \tag{30}$$

$$\rho = p = \frac{1}{8\pi G}\frac{6(3-\beta)}{\{(3-\beta)t + 2c_1\}^2}, \tag{31}$$

and

$$\Lambda = \frac{12\beta}{\{(3-\beta)t + 2c_1\}^2}. \tag{32}$$

The expansion scalar Θ and density parameter Ω are given by

$$\Theta = \frac{6}{(3-\beta)t + 2c_1}, \tag{33}$$

and

$$\Omega = \frac{3}{2}(3-\beta). \tag{34}$$

The deceleration parameter q is defined as $q = -1 - \frac{\dot{H}}{H^2}$. Thus, in this case, we have the following

$$q = \frac{1}{2}(1-\beta). \tag{35}$$

Equations (34) and (35) together give

$$\Omega = 3(q+1). \tag{36}$$

From equations (29) and (30) of the physical parameters of the model, we observe that the scale factor $a(t)$ and the spatial volume V are zero at $t = t_2$, where $t_2 = -2c_1/(3-\beta)$. It follows that the universe under the considered model has a finite-time big-bang singularity at $t = t_2$ which shifts to initial singularity by setting $c_1 = 0$. Here the Hubble parameter diverges at the time $t = t_2$. Moreover, the expansion scalar Θ, isotropic pressure p and matter density ρ all diverge at the time $t = t_2$. Also, in the limiting case for large t, all the physical parameters $H(t)$, Θ, p and ρ converge to zero where as the scale factor $a(t)$ and the volume V become infinite. Thus the considered model in this case represents a cosmological scenario in which the universe starts from a finite-time big-bang singular state and expands with cosmic time t. From equation (35) we observed that for $\beta < 1$ the value of the deceleration parameter q is positive which indicates the decelerating phase of expansion of the observed universe. This is mainly responsible for structure formation [11]. For $\beta > 1$ the current model indicates the accelerating phase of expansion. Recent observations favor accelerating models but they do not rule out decelerating models which are also consistent with present day observations.

4 Some Cosmological Parameters

4.1 Cosmological red-shift

When an object goes away radiating signals(light or sound), frequency of signals decreases and the wavelength increases. In the visible region of a spectrum of light, red has the highest wavelength and blue has lowest wavelength. So, in case, the wavelength of pulses increases and we have red-shift. When wavelength decreases, we have blue shift. Red-shift is a very useful parameter of observational cosmology. It gives spectral shift in wavelength, when a source of radiation moves away. Now we discuss red-shift in the cosmological context for conharmonically flat FRW model. Light travels from a galaxy to us. As the universe expands the wavelength of the light stretches in proportion to the amount of expansion of the universe. This means that if λ_e is the wavelength of the emitted light and λ_0 is the wavelength of the light we observe, then

$$\frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)}, \tag{37}$$

where t_e is the time at which the galaxy emitted the light. Red-shift is quantified by the relative change in wavelength z, which is expressed as

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e}. \tag{38}$$

From equations (37) and (38), we obtain

$$1 + z = \frac{a(t_0)}{a(t_e)} = \frac{a_0}{a}. \tag{39}$$

4.2 Statefinder parameters

Statefinder parameters $\{r, s\}$ play a significant role to describe the dynamics of the Universe in modern cosmological model. In this section we present the definition of the statefinder parameters and calculate them for conharmonically flat FRW model. It is to be noted that statefinder parameters solved the problem of discriminating between the various candidates of dark energy model. It is natural to look whether the above mentioned models agree with cosmological constant cold dark matter(ΛCDM) model which is considered as one of the most acceptable model in the present day observations. In an endeavor to discuss the cosmological scenarios of the universe Sahni [12] and Alam et al. [13] have introduced a pair of dimensionless geometrical parameters $\{r, s\}$ popularly known as Statefinder parameters. For a (ΛCDM) model, the Statefinder parameters $\{r, s\}$ have the value $\{1, 0\}$. The Statefinder parameters may be defined as follows:

$$r = \frac{\dddot{a}(t)}{a(t)H^3(t)}, \tag{40}$$

and

$$s = \frac{r-1}{3(q-\frac{1}{2})}, \tag{41}$$

where $a(t)$ is the expansion scale factor, $H(t)$ is the Hubble's parameter, q is the deceleration parameter and over head dot indicates differentiation with respect to the cosmic time t. The parameter r is also named as jerk parameter or as jolt or as super acceleration.

Hence for matter dominated considered model, we have the statefinder parameters as

$$r = (\beta-2)(2\beta-5), \tag{42}$$

and

$$s = \frac{(\beta-2)(2\beta-5)-1}{3(\frac{3}{2}-\beta)}. \tag{43}$$

For $\beta = 3$, equations (42) and (43) yield $r = 1$, $s = 0$. Thus, we obtain $\{r, s\} = \{1, 0\}$ which shows that the model is in agreement with ΛCDM model.

4.3 Cosmic snap parameter

The cosmic snap parameter plays a key role in cosmology to obtain the models like ΛCDM. The present day value of cosmic snap parameter may be used to characterize the evolutionary status of the universe. This parameter is defined as [14, 15]

$$s^\star = \frac{1}{a(t)H^4}\frac{d^4a(t)}{dt^4}. \tag{44}$$

The cosmic snap parameter is dimensionless and we can write

$$a(t) = a_0\{1 + H_0(t-t_0) - \frac{1}{2}q_0H_0^2(t-t_0)^2 + \frac{1}{6}r_0H_0^3(t-t_0)^3 + \frac{1}{24}s_0^\star H_0^4(t-t_0)^4 + O[(t-t_0)^5]\}, \tag{45}$$

where subscript O denotes the present day value of the concerned quantity. Equation (44) can also be represented as

$$s^\star = \frac{\dot{r}}{H} - r(2+3q), \tag{46}$$

where q is the deceleration parameter and r is the jerk parameter. The cosmic snap parameter (the fourth time derivative) is also sometimes called $jounce$. Since in the considered model $r = 1$, then equation (46) yields $s^\star = -(2+3q)$ and consequently $\frac{ds^\star}{dq} = -3$, which implies a measure of the evolution of the universe deviating from ΛCDM model [16, 17]. For the present matter dominated conharmonically flat FRW model $\beta = 3$ implies $q = -1$. Consequently equation (46) gives $s^\star = 1$. Thus the considered model has good similarity with the recent day observations.

4.4 *Om* parameter

In 2003, Sahni et al. [12] proposed a new cosmological parameter named Om which was introduced to differentiate ΛCDM from other dark energy models. The Om diagnostic method is actually a geometrical diagnostic which combines Hubble parameter and red-shift. Sahni and his collaborators demonstrated that irrespective to matter density content of universe acceleration probe can discriminate various dark energy models [18]. Om parameter is defined as follows:

$$Om(z) = \frac{[H(z)/H_0]^2 - 1}{(1+z)^3 - 1}.$$

Phantom like dark energy corresponds to the positive slope of $Om(z)$ whereas the negative slope means dark energy behaves like quintessence [19]. Also growth $Om(z)$ at late time favors the decaying dark energy models [20].

5 Conclusion

In the present paper we have investigated conharmonically flat spacetimes in the framework of FRW cosmological model. We have considered here perfect fluid as a source of matter distribution of the universe. Einstein's field equations with variable cosmological constant are solved by using special variation law $\Lambda = \beta H^2$, where β is a constant. These solutions describe a cosmological scenario in which the universe in the present model has finite-time big-bang singularity at some finite times $t = -c_1/(3-\beta)$ and $t = -2c_1/(3-\beta)$ for matter dominated solution and stiff era respectively. Also we observed that the universe expands with cosmic time t. In this case we found that Hubble parameter H, isotropic pressure p, energy density ρ, expansion scalar Θ and cosmological constant Λ all diverge at certain time with scale factor $a(t)$ and spatial volume V zero and the parameters $(p, \rho, H, \Theta, \Lambda)$ all the infinite whereas all these parameters $(p, \rho, H, \Theta, \Lambda)$ converge to zero as t tends to infinity with V and $a(t)$ infinite. These facts are in accordance with the present day observations. We also observe that the estimated values of Statefinder parameters $\{r, s\}$ and cosmic snap parameter $s^\star$ are agree with ΛCDM model. Moreover, we characterize red-shift parameter and Om parameter in conharmonically flat FRW spacetime model. Also different values of deceleration parameter indicates that the universe goes through deceleration and acceleration phase of expansion. These phenomena are consistent with the present day observations. Thus the present model has good consistency with recent cosmological studies and observations that make the model more objective.

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