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Muonic Kaon and the Kaon Size

David A. Owen*

Ben Gurion University of the Negev, Israel
*Corresponding author E-mail: daowen137@gmail.com

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Abstract. It is shown, by utilizing the energy levels determined by the Bethe-Saltpeter equation which includes recoil and retardation effects for a scalar fermion bound state which have been calculated the to α^4th [1] for the μ^--K^+ atom, the physical size of the charged K-meson can be determined.

KEY WORDS: k-muon size, quantum bound state of fermion and scalar particle.

1 Introduction

There is strong indication that the kaonium 2p state has been experimentally observed [2]. The interest in such a system is that the strong interaction can be treated as a perturbation as the kaon and anti-kanon are bound by electromagnetic interaction and that being bound in a p-state the kaon and anti-kaon are separated as opposed to an S-state. Thus the strong interaction, having a much shorter range than the separated kaons is a residual effect, in other words, a perturbation. Yet the individual structure of each kaon which affects the bound state energy level, is not known. It's size, how much space does it occupy, etc. The main idea of this work is that we are able to calculate via QED the energy levels for a relativistic system comprised of two charged scalar particles, with recoil corrections, retardation corrections etc. [1]. If the results obtained here are compared the observed $2p$ state of kaonium, the difference would be due to the finite size of the kaon and the residual strong interaction between the quark constituents of each of the kaons interacting with each other. By knowing precisely the the QED contribution, the discrepancy between the QED calculated value, would put a bound on the finite size contribution of the kaons and residual interaction been the two kaons, due to the fact, considering the $2p$ state, the kaon and anti-kaon are physically separated. In the Introduction, the relevant theory will be reviewed. However, there is a fundamental difficulty proceeding this way. If the size of the kaon is unknown, if the kaon and anti-kaon overlap physically, the quarks in the kaon and anti-kaon could be interacting which brings in the strong interactions. Instead, to deal with only QED interactions, a more reasonable system to consider is the bound K^+ with a μ^-. The mass of the K^+ is 493.667 MeV/c^2 roughly half the mass of the proton yet there is no idea what the size of a kaon is while the proton is about 0.85 fm. Since the kaon is a two quark system while the proton is a three quark system, one has no way of estimating the kaon size. Also worthwhile ago mention, the mass of μ is approximately 105.7 MeV.

As a motivation, we recall how the size of the proton is determined. Two methods are employed. One is method is to scatter beam of electrons off hydrogen and the second method is by spectroscopy. This first method is not relevant to our present considerations as a kaon target for scattering doesn't exist. The second method, described in detail by Repko and Dicus [3] (and reference cited therein) become more precise if one considers muonic hydrogen instead of ordinary hydrogen as the muon being closer to the proton is more sensitive to the proton size.

2 Proton Size from Spectroscopy

As this is the method we would use, we describe it briefly. They [3] utilize QED potentials derived from the two-particle field equation describing a bound state known as the Bethe-Saltpeter equation [6]. This equation includes recoil and radiation corrections to the bound state. But, this is assuming that each of the particles comprising the bound state are point like. The effect of the proton size can be obtained in terms of mean square radius $\langle r^2 \rangle$ by modifying the Coulomb potential with charge form factor $F(\mathbf{k}^2)$ defined as

$$F(\mathbf{k}^2) = \int d^3 r \rho(r) e^{-i\mathbf{k}\cdot\mathbf{r}} \,, \tag{1}$$

where $\rho(r)$ is the proton charge density. In momentum space the Coulomb potential becomes

$$V(\mathbf{k}^2) = -e^2 \frac{F(\mathbf{k}^2)}{\mathbf{k}^2} \,. \tag{2}$$

Expanding the exponent in Eq. 1 and integrating gives

$$F(\mathbf{k}^2) = 1 - \frac{1}{6}\langle r^2 \rangle + \cdots \,, \tag{3}$$

where $V(k^2)$ is the Coulomb potential in momentum space and $F(k^2)$ is the corresponding form factor.

Thus in momentum space, $V(\mathbf{k}^2)$ can be written

$$V(\mathbf{k}^2) = -\frac{\alpha}{\mathbf{k}^2} + \frac{\alpha}{6}\langle r^2 \rangle + \cdots \,. \tag{4}$$

Writing Eq. 4 in coordinate space, we have

$$V(r) = -\frac{\alpha}{r} + \frac{2}{3}\pi\alpha\langle r^2 \rangle \delta(\mathbf{r}^2) \,. \tag{5}$$

This leads to a perturbative energy due to the finite size of the proton

$$\Delta E = \frac{2}{3}\frac{\mu^3\alpha^4\langle r^2\rangle}{n^3}\delta_{l0}\,. \tag{6}$$

In the work we have cited [3] they compare the $2^1S_{1/2} \leftrightarrow 2^3P_{2/3}$ interval calculated theoretically with that obtained experimentally

$$229.5652\ \mathrm{Mev} - 5.2257r_p^2\ \mathrm{MeV\,fm}^{-2} + (0.035r_p^3) = 225.8535\ \mathrm{Mev}\,,$$

where $r_p = \sqrt{\langle r^2\rangle}$.

Solving for $r_p = 0.8428$ fm (0.8452 fm).

The crucial point to note is that using the observed difference in the spectrum of hydrogen and the above result to determine to determine $\langle r^2\rangle$ gives essentially the same result as obtained by scattering electrons off hydrogen. This is important in that in the current proposal only the spectral result is available but should be considered reliable.

Therefore, perhaps we can take over the theoretical calculations given in [3]. There's a glitch as will be explained. The theoretical expression given in [3] were calculated by the quantum field equation for the bound state, viz (the Bethe-Saltpeter equation) which we shall show inapplicable for the system we are considering. To understand the why their theoretical method cannot be used in the kaon-muon system we briefly review the quantum field theory tor bound states.

3 Quantum Field Theory for Bound States

The quantum field theory for two particle bound state was developed by Bethe and Saltpeter [6]. They assumed that the interaction between the two particles could be described by Feynman diagrams. By selective summing the diagrams they obtained what is referred to as the Bethe-Salpeter equation:

$$\begin{aligned} G(x_3,x_4,x_1,x_2) = S_F^{1'}(x_3,x_1)S^{2'}(x_4,x_2) + S_F^{1'}(x_3,x_5)S^{2'}(x_4,x_6) \\ \times I(x_5,x_6;x_7,x_8)G(x_7,x_8;x_1,x_2)\,, \end{aligned} \tag{7}$$

where $G(x_3,x_4,x_1,x_2)$ is the two particle propagator; $S_F^{1'}(x_3,x_1)$ – full one particle propagator; $I(x_5,x_6;x_7,x_8)G(x_7,x_8;x_1,x_2)$ is the irreducible two particle kernel.

Due to the generality of the derivation of the Bethe-Saltpeter equation, the Feynman expansion off the two particle propagator into a set in which the particles don't interact and the other set in which they do. The boundary condition for a bound state is to ignore the diagrams for which the particles don't interact the remaining set of diagram constituents the Bethe-Saltpeter equation for a bound state. one would think it would apply to any two particle bound state. For this to be the case, it should have a non-relativistic reduction to the Schrödinger equation. for the case ot two bound fermions, Bethe and Saltpeter demonstrated how this is done and from this followed a sensible perturbation theory. This is crucial as the Bethe-Saltpeter equation of events simplest realist solutions has no-known exact solutions. Thus it is relating the Bethe-Saltpeter equation to at the non-relativistic Schrödinger equation as a basis for beginning a perturbation expansion.

3.1 Perturbation theory of Bethe Saltpeter equation

The bound state equation in QED which describes the bound state of a two particle system was found by both Bethe & Saltpeter, J. Schwinger & M. Gell-Mann and F. Low. This equation, for fermion constituents, can be written as

$$\psi(x_1,x_2) = S_F^{(1)}(x_1 - x_1')S_F^{(2)}(x_2 - x_2') \times I(x_1', x_2'; x_3, x_4)\psi(x_3, x_4)\,, \quad (8)$$

where the self-interactions of the fermion propagators have been absorbed in the interaction kernel $I(x_1', x_2'; x_3, x_4)$ and the two particle wave function is given by ψ which is a function of each of the particles coordinates and the S_F^i is the fermion propagator for the i-th particle.

Schematically, Eq. 8 can be written

$$\phi_K(x) = G_K(x,x')I_K(x',x'')\phi_K(x'')\,, \quad (9)$$

where $\phi_K(x) = \psi(x_1, x_2)$ with $x = x_1 - x_2$ and

$$G_K(x,x') = \int \frac{d^4p}{(2\pi)^4} \exp[-ip\cdot(x-x')] \times \frac{1}{[(\eta_a K + p)\gamma^{(a)} - m_a][(\eta_b K - p)\gamma^{(b)} - m_b]} \quad (10)$$

for two particles of mass m_a and mass m_b with $\eta_a = m_a/(m_a + m_b)$ and $\eta_b = m_b/(m_a + m_b)$, K is the bound state energy while $I_K(x', x'')$ is the interaction kernel.

Since Eq. 9 cannot be solved exactly, following Saltpeter [4] and separate Eq. 10 by partial fractions, i.e.

$$\frac{1}{[(\eta_a K + p)\gamma^{(a)} - m_a][(\eta_b K - p)\gamma^{(b)} - m_b]} = \frac{1}{K - H^{(a)}(\mathbf{p}) - H^{(b)}(-\mathbf{p})} \times \left\{ \frac{1}{\eta_a K + p_0 - H^{(a)}(\mathbf{p} + i\epsilon)} + \frac{1}{\eta_b K - p_0 - H^{(b)}(-\mathbf{p} + i\epsilon)} \right\} \gamma_0^{(a)}\gamma_0^{(b)} \quad (11)$$

with $H^{(a)}(\mathbf{p}) = \alpha \cdot \mathbf{p} + \beta m_a$ and $H^{(b)}(-\mathbf{p}) = -\alpha \cdot \mathbf{p} + \beta m_b$.

Equation 9 can now be written

$$[K - H^{(a)}(\mathbf{p}) - H^{(b)}(-\mathbf{p}]\phi_K(x) = \Lambda_K(x,x')\gamma_0^{(a)}\gamma_0^{(b)} I_K(x',x'')\phi_K(x'') \quad (12)$$

with

$$\Lambda_K(x,x') = \int \frac{d^4p}{(2\pi)^4} \exp[-ip \cdot (x - x')] \times \left\{ [\eta_a K + p_0 - H^{(a)}(\mathbf{p})]^{-1} + [\eta_b K - p_0 - H^{(b)}(-\mathbf{p})]^{-1} \right\}. \quad (13)$$

Equation 12 is from where a perturbation, for a system of two fermions could begin. If we were to separate out of I_K the Coulomb interaction, we can recover Schrödinger equation with the Coulomb interaction. However, if in Eq. 11 one of the propagators replaced by a scalar propagator, representing the K-meson, all the reduction above, which lead to theoretical value to evaluation the QED contribution for the μ proton system cannot be done and is irrelevant for the kaon-muon system. The partial fraction separation relied on that the inverse of each of the propagators were linear in their momentum which isn't true if one or both particles have integer spins.

Specifically, if we replace one of the propagators in Eq. 11 we would have

$$\frac{1}{[(\eta_a K + p)\gamma^{(a)} - m_a][(\eta_b K - p)^2 - m_b^2]}. \quad (14)$$

Clearly Eq. 14 doesn't cannot be separated by partial fractions as was the case when both particles were fermions.

4 Kaon-Muon Bound State

As we have mentioned, the foregoing bound state considerations applies to fermion-fermion bound states and the perturbation theory described is not applicable. A systematic approach to the scalar-fermion and scalar-scalar bound state was developed by Owen [1] in 1994. For our present considerations, the result for the fermion-scalar α^4 contribution will cited as it will be sufficient for the following:

$$\Delta E(\alpha^4)_{0-1/2} = -\frac{\mu^4\alpha^4}{2n^4}\left(\frac{1}{m_a^3} + \frac{1}{m_b^3}\right)\left(\frac{n}{l+1/2} - \frac{3}{4}\right) + \frac{\mu^3\alpha^4}{m_a m_b n^4} - \frac{3\mu^3\alpha^4}{2m_a m_b}\frac{1}{n^3(l+1/2)} + \frac{\mu^3\alpha^4}{n^3}\left(\frac{1}{2m_a^2} + \frac{1}{m_a m_b}\right) \times \left\{\frac{(1-\delta_{l0})}{l+1}\left(\frac{\theta[j-(l+1/2)]}{2l+1} - \frac{\theta[j-(l-1/2)]}{l}\right)\right\} + \delta_{0l}. \quad (15)$$

For the $2S \leftrightarrow 2P$ transition in a muon-kaon atom

$$\Delta E(\alpha^4)_{0-1/2}(l=0, j=1/2, n=2) + \frac{2}{3}\frac{\mu^3\alpha^4\langle r_K^2\rangle}{8} - \Delta E(\alpha^4)_{0-1/2}(l=1, j=3/2, n=2) = \text{experimental value}\,, \quad (16)$$

where we take $m_a = 105.6580$ MeV, $m_b = 493.677$ MeV, $\mu = 87.0296$ MeV. Or more explicitly we can from the above equation, Eq. 16 that the mean square radius of the K^+, $\langle r_K^2\rangle$ is

$$\langle r_K^2\rangle = \frac{12\alpha^{-4}}{\mu^3}\Big\{\Delta E(\alpha^4)_{0-1/2}(l=1, j=3/2, n=2) - \Delta E(\alpha^4)_{0-1/2}(l=0, j=1/2, n=2)\Big\} - \text{experimental measured energy for the } 2P_{3/2} - 2S_{1/2} \text{ transition in the } K^+ - \mu^- \text{ atom.} \quad (17)$$

References

[1] D.A. Owen (1994) *Found. Phys.* **24** 273.
[2] R. Lichard (2020) *Phys. Rev.* **101** 111501.
[3] W. Repko, D. Dicus (2018) *Phys. Rev. D* **98** 013002.
[4] E.E. Saltpeter (1952) *Phys. Rev.* **87** 328.
[5] T. Fulton, D.A. Owen, W.W. Repko (1971) *Phys. Rev. A* **4** 1802.
[6] E. Saltpeter, H. Bethe (1951) *Phys Rev.* **84** 1232;
J. Schwinger (1951) *Proc. Natl. Acad. Sri. USA* **37** (1951);
M. Gell-Mann, F. Low (1951) *Phys. Rev.* **84** 350.
[7] R.D. Peccei, Helen Quinn (1977) *Phys. Rev. Lett.* **38** 1440.
[8] D.A. Owen (1990) *Phys. Rev. D* **42** 3534.
[9] D.A. Owen (1977) *Phys. Rev. A* **16** 1594.
[10] H. Bethe, R. Jackiw (1968) *Intermediate Quantum Mechanics*. The Benjamin/Cummings Publishing Company, London.